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J. W. L. GLAISHER, Sc.D., F.R.S.,

FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

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CONTENTS OF VOL. LIII.

MESSENGER OF MATHEMATICS.

ON CERTAIN PUZZLE-QUESTIONS OCCURRING IN EARLY ARITHMETICAL WRITINGS AND THE GENERAL PARTITION PROBLEMS WITH WHICH THEY ARE CONNECTED.

By *J. W. L. Glaisher*.

[In this paper, apart from historical matter, the subjects treated of are the partitionment of a finite series of numbers in arithmetical progression, when one of the numbers is given and also the common difference, into (i) numbers of the form $p\alpha+\beta$ where $\alpha+\mu\beta$, p, and μ are the same for each of the numbers in the series, (ii) into $\alpha+\beta$ where $\lambda\alpha+\mu\beta$, λ, and μ are the same for each of the numbers in the series, and (iii) into $p\alpha+q\beta$ where $\alpha+\beta$, p, and q are the same for each of the numbers in the series.]

Part I.

Generalisation of Widman's question. Systems of numbers which when divided by p are such that the sum of the quotient and μ times the remainder is the same, p and μ being given numbers.

On a certain puzzle-question, §§ 1—3.

§ 1. One of the enigmata or puzzle-questions which occur in some of the early manuscripts and printed books is that of three women who sell different numbers of eggs at the same price and yet bring back the same money. This question and solution may be stated as follows: they have respectively 50, 30, and 10 eggs which they sell at 7 a penny; thus the first receives 7 pence and has 1 egg over; the second receives 4 pence and has 2 eggs over; the third receives 1 penny and has three eggs over: the eggs left over are then sold at

3 pence each, so that altogether each woman takes home 10 pence.*

§ 2. The question occurs in a 14th century manuscript in the Munich Library (Cod. lat. Monac. 14684, ff. 30–33), which was printed by Curtze in the *Bibliotheca Mathematica*† for 1895. The manuscript begins "*Incipiunt subtilitates enigmatum*", and the thirteenth enigma is "*Item alia subtilitas.* Quidam committens filiis suis tribus pira vendere dicit seniori: exeas cum 50, et vendas prout melius possis, et reporta precium. Post hoc alteri 30, juniori vero 10, et iniunxit utrisque iunioribus filiis, quod omnimode sicut senior venderent sua pira et tot pro denario, et reportarent tantam pecuniam, quantam primus. Senior vero exiens vendidit 7 pro denario, et unum pirum, quod remansit pro tribus denariis. Secundus vero considerans diligenter vendicionem senioris vendidit quater 7 pro denario, et 2 pira remanencia pro 6 denariis, et reportabat 10 denarios sicut primus. Tercius autem dedit 7 pro denario et tria remanencia pro 9 denariis vendidit, et similiter reportabat patri suo 10 denarios sicut fratres sui".

Rath states that the question is found in a Vienna manuscript Codex Vindob. 3029 (of about 1480), and a Stuttgart manuscript of 1488.‡

It is given by Widman on p. 134′ of his *Rechenung* (1489),§ where it is followed by some developments which gave occasion to the present paper.

§ 3. The question occurs also in Tagliente's *Libro de Abaco* ‖ (1515), where a reason is given to justify the increased price at which the eggs that are left over are sold. The question and solution are: "Do me this question. Three women go to market to sell eggs, and one carries 50 eggs, the second 30, the third 10, and all sell at the same price, and all carry home the same money. I ask at what price they sell."

* The present paper was originally intended to have been included (in a briefer form) in the paper "On the early history of the signs + and − and on the early German arithmeticians" (*Messenger of Mathematics*, vol. li., pp. 1–148), as mentioned in the note on p. 131.

† Ser. 2, vol. ix., pp. 77–88. In the *Zeitschrift für Math. und Phys.*, vol. xl., supp., p. 35, note, Curtze refers to this manuscript as of the 13th century.

‡ *Bibl. Math.*, ser. 3, vol. xiv., p. 247.

§ "Behēde vnd hubsche Rechenung auff allen kauffmanschafft" (Leipzig, 1489). The later editions are given on p. 5, vol. li., of the *Messenger*, in the paper referred to in the first note on this page. That paper also contains (pp. 146–148) the titles of most of the books and papers mentioned in Part I. of the present paper, with reference to the places in that paper where fuller details are given.

‖ The full title of this work, and of the editions of 1525 and 1527, as well as an account of their relations to one another, are given in the notes on pp. 80, 81, 82 of vol. li.

"This is the rule. You say these women begin to sell these eggs at 7 for a soldo, and the one who has 50 eggs will have 7 soldi and 1 egg will remain; the one who has 30 sells them for 4 soldi and 2 remain; the one with 10 sells them for 1 soldo and 3 eggs remain. And it happened that there were no other eggs in the place, and one came who had great need of them, and took all the eggs that were left at 3 soldi each, so that the one who at first had 7 soldi for 49, had 3 soldi for 1 egg, which makes 10 soldi; and the second who had 4 soldi for 28, had 6 soldi for the 2 eggs, which makes 10 soldi; and the third who had 1 soldo for 7, had 9 soldi for the 3 eggs, which makes 10 soldi. So that the question is right, and you will do similar questions in this way".

The question is repeated in nearly the same form in the edition of 1527: but in the edition of 1525 it takes the form: "There are three women who go to market to sell eggs: the first carries 20, the second 40, and the third 60. And the three women all sell at one price, and at the end find that they have all taken the same number of soldi. I ask at what price each one has sold hers. Do thus. Know that when these women had come to market, each one sold 7 eggs for a soldo, so that the first has 2 soldi for her eggs and has 6 eggs left, and the second has sold her eggs for 5 soldi and has 5 eggs left, and the third has sold her eggs for 8 soldi and has 4 eggs left. Then the first sells the 6 eggs left at 3 soldi each which is 18 soldi and 2 soldi at first make 20 soldi. And the second has 5 soldi for hers and 5 eggs left, which at 3 soldi each amounts to 15 soldi and 5 soldi at first make 20 soldi. Then the third who has 4 eggs left at 3 soldi each has 12 soldi for hers and 8 soldi at first make 20 soldi. And it will be done".*

Widman's treatment of the question (1489), §§ 4–7.

§ 4. Widman's statement and explanation of the question are as follows: †

"Apfel

"Itm̄ eyn pawer hat 3 tochter Vnnd giebt der ersten 10 Apfel Der andern 30 Vnd der drittn̄ 50. vn̄ sol ye eyne als vil pro 1ꝯ gebn̄ alss die ander Nu ist die frag wie vil sol itliche pro 1ꝯ gebn̄. vnd wivil kost itliche geltz Machss also vnd sprich das

* In this question the numbers of the eggs are 60, 40, 20. In the 14th century manuscript, in Widman, and in the 1515 and 1527 editions of Tagliente they are 50, 30, 10. I do not know whether they are 50 30, 10 in the three manuscripts mentioned by Rath. In Blasius, referred to in § 25, the numbers are 8, 17, 26.

† pp. 134′–135.

itliche 7 apfel giebt pro 1₰ Vn̄ was dan mynner ist dan 7. dan giebt sy ye 1 apfel pro 3₰ Nu machss vnd secz alszo.

Nu seȳ	1	7	3	10
yr drey	4	7	2	30
Darūb	7	7	1	50

3 von 7 pleybn̄ 4 Die secz darnach nym 3 von 4 pleybt 1 wiltu nu habn̄ die apfel Multiplicir 1 mit 7 vn̄ addir 3 facit 10 dy erst Darnach multiplicir 4 mit 7 addir 2 facit 30 Die ād̄er darnach multiplicir 7 mit 7 addir 1 facit 50 Die dritte Nu weystu dy apfel vn̄ wilt wissen wie vil itliche gebn̄ szol pro 1₰ So subtrahir 3 von 10 pleyben 7 das diuidir mit 1 facit 7 apfel pro 1₰ Nu des gleichen nym 2 von 30 vnnd diuidir mit 4 facit 7 Auch nym 1 vonn 50 pleibt 49 das teyl in 7 facit 7 apfel Vnd is gemacht".

§ 5. The meaning of the numbers in the table is evident: the product of the first two numbers in each line added to the third gives the number of eggs: and three times the number in the third column added to that in the first gives the sum received. If we know that 7 is the number of eggs sold for a penny, and that the price of each of those left over is 3 pence, and we also know the number of eggs each woman had, then each line of the table can be at once constructed separately: but the reason for the way in which Widman forms it is not clear, for he derives the first and second numbers 1, 4 in the first column from the lowest number 7 by repeated subtraction of 3, which is the first remainder, and also the price of each egg left over, and also the number of the women: and it is not obvious why he should have begun with the lowest number. More light, however, is thrown on this matter by three other similar questions which Widman gives immediately afterwards.

§ 6. These questions are*

"Apfel

"Itm̄ 3 tochter als obn̄ die sollen gebn̄ 9 apel pro 1₰ &c.

3	9	3	30	Nu secz
6	9	2	56	also
9	9	1	82	Mul

* pp. 135–136.

tiplicir 3 mit 9 Vnd addir 3 dar zu facit 30 Die erst tochter Darnach multiplicir 6 mit 9. vnd addir 2 facit 56 Die ander Darnach multiplicir 9 mit 9. vñ addir 1 facit 82 Die dritt Wiltu nu wissen wy vil apel Szo nym 3 von 30 vñ diuidirss durch 3 facit 9 Vñ alszo mach auch dy andern

1	13	4	17	Itm̄
5	13	3	68	4 toch
9	13	2	119	ter &c.
13	13	1	170	Vnd

itliche sol gebñ 13 apel pro 1ϑ &c. Secz also. Wiltu habñ die apel so subtrahir 4 vom 17 vnd diuidirss durch 1 facit 13 &c."

"Itm̄ eyñ pawer hat 5 tochter &c. wȳ vor

10	30	5	305
15	30	4	454
20	30	3	603
25	30	2	752
30	30	1	901

"Ist die frag wie vil itliche apel geb. vnd wie vil hat itliche apel &c. Nu seyn do der tochter 5 Darumb sollñ sy ye 1 apel der vberig ist pro 5ϑ gebñ Wan sso yr 4 seyn szo gebñ sy 1 vberigen apel pro 4ϑ &c. vnd machss wie obñ".

§7. Widman's diagrams and explanations suggest that he (or a previous writer whom he followed) was desirous of forming questions of the same type as that in which the numbers of eggs were 10, 30, 50; and for this purpose he gives a method of procedure when the number of women was given, and also the number of eggs that were sold for a penny: thus for 4 women, and if the eggs were to be sold at 13 for a penny, he would first construct the scheme

		4
		3
		2
13,	13,	1

and then fill in the first column from the bottom upwards by

the continued subtraction of 4, the top figure in the last column (which is also the number of women), thus giving 9, 5, 1, and fill in the second column by repeating the 13. In this manner, if the number of women and the number of eggs to be sold for a penny were given, he could assign to each woman a number of eggs which would enable them all to bring home the same money, and by his procedure each egg left over would cost as many pence as there were women.

If there were r women, and they sold their eggs at p for a penny, the lowest line in the table would be p, p, 1, and the right-hand column, starting from the bottom, would be 1, 2, ..., r; the second column would consist wholly of p's; and the first, starting from the bottom, would be p, $p-r$, $p-2r$, ..., $p-(r-1)r$. Then the first woman would have $p^2-(r-1)pr+r$ eggs, the next one $pr-1$ more eggs, the next $pr-1$ more eggs still, and so on, the last having p^2+1 eggs: and the number of pence received by each woman would be $p+r$.

Before, however, examining further Widman's treatment of the question, it is interesting to consider the general theory.

The general theory: various generalisations in which the sum of the quotient and μ times the remainder is the same, §§ 8–11.

§ 8. Regarded arithmetically, the general question suggested by the original problem may be stated in the form: find a system of numbers such that when divided by the same divisor the quotient $+\mu$ times the remainder is the same for all; or, in other words, if p and μ are any two given numbers, find a system of numbers such that when divided by p, the quotient $+\mu$ times the remainder is the same for all.

If $n=ap+b$ and $n'=a'p+b'$ and $a+\mu b=a'+\mu b'$, it follows that $a'-a=\mu(b-b')$. Therefore $a'-a$ must be divisible by μ, and we have

$$n'-n=(b-b')(p\mu-1)=\frac{a'-a}{\mu}(p\mu-1).$$

Thus the difference between any two numbers in the system must be a multiple of $p\mu-1$, and the difference between two quotients must be a multiple of μ and be equal to μ times the difference between the remainders.

The longest sequence of numbers having the required property is obtained by taking the difference between consecutive remainders to be unity and making the first remainder as large as possible, viz. $p-1$. We thus obtain the following table, in which the columns give the quotient, divisor, remainder, and multiplier:

$$\left|\begin{array}{llll} a & ,\ p: & p-1; & \mu \\ a+\mu & ,\ p: & p-2; & \mu \\ \dots & & & \\ a+(p-2)\,\mu, & p: & 1\ ; & \mu \\ a+(p-1)\,\mu, & p: & 0\ ; & \mu \end{array}\right| .$$

The numbers which are represented by the system are obtained by multiplying the numbers in the first two columns and adding that in the third, and the fixed-sum* is obtained by adding to the number in the first column the product of the numbers in the last two. Thus the numbers forming the system begin with $pa+p-1$ and increase by $p\mu-1$, the last being $pa+p-1+(p-1)(p\mu-1)$, that is, $pa+p(p-1)\mu$; and the fixed-sum is $a+(p-1)\mu$. The quotients in the first column increase by μ, and the remainders in the third column decrease by unity.

§ 9. The simplest and most interesting case is that in which a is taken to be equal to $\mu-1$. The first number of the system, the common difference of the numbers in the system, and the fixed-sum are then all equal to $p\mu-1$. This system, which consists of p numbers, may be written

$$\begin{array}{r|rlll|l} p\mu-1 & \mu-1, & p: & p-1; & \mu & p\mu-1 \\ 2(p\mu-1) & 2\mu-1, & p: & p-2; & \mu & ,, \\ \dots & \dots & & & & \dots \\ (p-1)(p\mu-1) & (p-1)\mu-1, & p: & 1\ ; & \mu & ,, \\ p(p\mu-1) & p\mu-1, & p: & 0\ ; & \mu & ,, \end{array} .$$

The system of numbers and the fixed-sum are written out side the table of quotients, &c., the four columns of which show the quotient, divisor, remainder, and multiplier.

§ 10. In the system in § 8 the first quotient a is arbitrary, but if the fixed-sum is given, a becomes determinate. Thus if p, μ are given and also σ the fixed-sum, then, if $\sigma>(p-1)\mu$, the system is as follows:

$$\begin{array}{l|llll|l} \sigma p-(p-1)(p\mu-1) & \sigma-(p-1)\mu, & p: & p-1; & \mu & \sigma \\ \sigma p-(p-2)(p\mu-1) & \sigma-(p-2)\mu, & p: & p-2; & \mu & ,, \\ \dots & \dots & & & & .. \\ \sigma p-(p\mu-1) & \sigma-\mu & ,\ p: & 1\ ; & \mu & ,, \\ \sigma p & \sigma & ,\ p: & 0\ ; & \mu & ,, , \end{array}$$

* For brevity of expression it is convenient to give the name 'fixed-sum' to the sum of the quotient and μ times the remainder.

The highest number is σp, and the other $p-1$ numbers are derived from it by continually subtracting $p\mu-1$.

If $\sigma<(p-1)\mu$, and if σ lies between $s\mu$ and $(s+1)\mu$, there are $s+1$ numbers in the system, which then is

$$\begin{array}{l|l|l} \sigma p - s(p\mu-1) & \sigma - s\mu \quad , p: s \quad ; \mu & \sigma \\ \sigma p - (s-1)(p\mu-1) & \sigma-(s-1)\mu, p: s-1; \mu & ,, \\ \cdots\cdots & \cdots\cdots & \cdots \\ \sigma p - (p\mu-1) & \sigma-\mu \quad , p: 1 \quad ; \mu & ,, \\ \sigma p & \sigma \quad , p: 0 \quad ; \mu & ,,. \end{array}$$

The numbers forming the system consist therefore of σp and the s numbers derived from it by continually subtracting $p\mu-1$.

Since $s=I\left(\frac{\sigma}{\mu}\right)$, where $I\left(\frac{l}{m}\right)$ denotes the largest integer in $\left(\frac{l}{m}\right)$, the first line may be written

$$\sigma p - I\left(\frac{\sigma}{\mu}\right)(p\mu-1) \;\Big|\; \sigma - I\left(\frac{\sigma}{\mu}\right)\mu,\ p: I\left(\frac{\sigma}{\mu}\right);\ \mu \;\Big|\; \sigma.$$

The numbers increase by $p\mu-1$ till σp is reached, the quotients increase by μ till σ is reached, and the remainders decrease by unity till 0 is reached.

§ 11. We may conveniently express these results by writing the lines of the system in the reverse order, so that the first line contains the largest number, the largest quotient, and the remainder zero; the two former decrease regularly by $p\mu-1$ and μ respectively, and the remainder increases by unity. The system thus written is

$$\begin{array}{l|l|l} \sigma p & \sigma \quad , p: 0; \mu & \sigma \\ \sigma p - (p\mu-1) & \sigma-\mu , p: 1; \mu & ,, \\ \sigma p - 2(p\mu-1) & \sigma-2\mu, p: 2; \mu & ,, \\ \cdots\cdots & \cdots\cdots & \cdots \\ \sigma p - s(p\mu-1) & \sigma - s\mu, p: s; \mu & ,, , \end{array}$$

where s is equal to $I\left(\frac{\sigma}{\mu}\right)$ if $\sigma \leqq (p-1)\mu$, and $p-1$ if $\sigma \geqq (p-1)\mu$.

The notation used in the representation of the systems, § 12.

§ 12. In the notation for the systems of numbers which I have used in § 8 and succeeding sections I have followed

Widman, merely adding another column giving the price at which the eggs were sold at the second sale: and I have separated the columns by a comma, a colon, and a semi-colon.

From the representation $|\alpha, p : \beta ; \mu|$ the number represented is obtained by multiplying the first two numbers, which are separated by a comma, and adding β, ignoring μ, which is separated from it by a semi-colon. The effect of the comma and semi-colon is reversed in the case of the fixed-sum, which is obtained by taking α and ignoring p and adding the product of β and μ.

Thus in forming the number a comma indicates union (by multiplication) and a semi-colon is a direction to reject the following number: while in the formation of the fixed-sum their meanings are reversed. The colon always indicates addition.

Thus the two representations

$$|\alpha, p : \beta\mu,\ \mu^{-1}| \text{ and } |\alpha p ; p^{-1} : \beta ; \mu|$$

are equivalent to $|\alpha, p : \beta ; \mu|$.

In interpreting a system with respect to the sale of eggs, a comma between two numbers, as in α, p, denotes that α pence are produced by the sale of eggs at p for a penny, which is equivalent to αp eggs being sold at $\frac{1}{p}$th of a penny each; and $\beta ; \mu$ indicates that β eggs were sold at μ pence each, which is equivalent to $\beta\mu$ pence being produced by selling $\frac{1}{\mu}$th of an egg for a penny.

Systems in which n and p are given, §§ 13—17.

§ 13. The question may also be asked in a somewhat different form, viz. to find a system of numbers which when divided by p have the same fixed-sum as a given number n, the multiplier μ being also given.

If $n = ap + b$, b being the remainder, the system of quotients, remainders, &c. is

$$\left|\begin{array}{lllll} a, & p : & b & ; & \mu \\ a+\mu, & p : & b-1 & ; & \mu \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots} \\ a+(b-1)\mu, & p : & 1 & ; & \mu \\ a+b\mu, & p : & 0 & ; & \mu \end{array}\right|$$

and also

$$\begin{array}{|l|}
a-\ \mu,\quad p:\quad b+1;\quad \mu \\
a-2\mu,\quad p:\quad b+2;\quad \mu \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array},$$

the latter portion of the table being continued so long as the quotients are positive and the remainders do not exceed $p-1$.

The numbers forming the system are therefore

$$n,\ n+p\mu-1,\ n+2(p\mu-1),\ \ldots,\ n+b(p\mu-1);$$

and also $\qquad n-(p\mu-1),\ n-2(p\mu-1),\ \ldots$

so long as these numbers are positive*, and the remainder does not exceed $p-1$. If in the descending branch, this remainder is reached while the numbers are still positive, the number of lines in this branch is $p-1-b$: but if the positive numbers are exhausted before the remainder $p-1$ is reached the number of lines is $I\left(\frac{a}{\mu}\right)$.

If $a \geqq (p-1-b)\mu$ the system contains the maximum number, p, of lines; but if $a<(p-1-b)\mu$ the number of lines is $b+1+I\left(\frac{a}{\mu}\right)$. Thus the number of lines in the system is either p or $b+1+I\left(\frac{a}{\mu}\right)$, whichever of the two is smaller.

We may also express this result conveniently by means of the fixed-sum σ, which $=a+\mu b$, viz. if $\sigma \geqq (p-1)\mu$, the system contains p lines, and if $\sigma<(p-1)\mu$ the number of lines is $1+I\left(\frac{\sigma}{\mu}\right)$, so that the number of lines is the smaller of the quantities p or $1+I\left(\frac{\sigma}{\mu}\right)$. This is also evident directly from the system in § 10.

It will be noticed that the largest quotient is equal to the fixed-sum and therefore the largest number in the system is p times the fixed-sum.

§ 14. As examples, let $p=5$, $\mu=2$; then if $n=33$, we have

$$\begin{array}{r|rrrr|l}
33 & 6, & 5: & 3; & 2 & 12 \\
42 & 8, & 5: & 2; & 2 & ,, \\
51 & 10, & 5: & 1; & 2 & ,, \\
60 & 12, & 5: & 0; & 2 & ,,
\end{array},$$

* It is evident that a number m and its corresponding quotient a become negative at the same time, for $m=ap+\beta$ where $\beta<p$; and therefore if m is negative, a must also be negative, and if a is negative, m must also be negative.

and the descending branch is

$$24 \mid 4, \quad 5: \quad 4; \quad 2 \mid 12;$$

and if $n=17$, we have

$$\begin{array}{r|llll|l} 17 & 3, & 5: & 2; & 2 & 7 \\ 26 & 5, & 5: & 1; & 2 & ,, \\ 35 & 7, & 5: & 0; & 2 & ,, \end{array},$$

and the descending branch is

$$8 \mid 1, \quad 5: \quad 3; \quad 2 \mid 7.$$

In the first example the descending portion ends (at 24) because the maximum remainder has been reached: in the second example it ends (at 8) because the least positive number in the system has been reached.

Applying the formula: in the first case $a=6$, $b=3$, $\mu=2$, so that $b+1+I\left(\frac{a}{\mu}\right)=7$, which $>p$, and there are p numbers in the system: in the second, $a=3$, $b=2$, $\mu=2$, so that $b+1+I\left(\frac{a}{\mu}\right)$ is 4, which $<p$, and there are 4 numbers in the system.

§ 15. The general system in § 8 shows that by taking $\mu=1$ we can always obtain a system of p numbers having $p-1$ as the common difference (p being any number) and in which the sum of the quotient and remainder is the same. This system may be written

$$\begin{array}{l|llll|l} pa+p-1 & a & , \; p: & p-1; & 1 & a+p-1 \\ pa+2(p-1) & a+1 & , \; p: & p-2; & 1 & ,, \\ \cdots\cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ pa+(p-1)^2 & a+p-2, & p: & 1; & 1 & ,, \\ pa+p(p-1) & a+p-1, & p: & 0; & 1 & ,, \end{array}.$$

If we take $a=0$, the first number, the common difference, and the fixed-sum are all the same.

Thus for any given divisor p there is always a system of p numbers for which the fixed-sum is $p-1$: and we can always obtain such a system of p numbers for any given fixed-sum greater than $p-1$. Taking as an example $p=11$

and $a=0$, we have the system of 11 numbers.

10	0, 11 : 10 ; 1	10
20	1, 11 : 9 ; 1	,,
...	....	...
100	9, 11 : 1 ; 1	,,
110	10, 11 : 0 ; 1	,, .

§ 16. In order to form a system of numbers having a given difference d, and which shall not consist of selected numbers from a system with a smaller difference, we must have $p\mu - 1 = d$, so that p must be a divisor of $d+1$, μ being its conjugate. Thus there are as many such systems as there are divisors of $d+1$, unity not being counted as a divisor. For example, if the common difference is 20, then $d+1$ is 21, and we have $p=21$, $\mu=1$; or $p=7$, $\mu=3$; or $p=3$, $\mu=7$. Taking the fixed-sum to be 20 also, we have the following three systems,* which contain respectively 21, 7, and 3 lines:

20	0, 21 : 20 ; 1	20,
40	1, 21 : 19 ; 1	,,
...	....	...
400	19, 21 : 1 ; 1	,,
420	20, 21 : 0 ; 1	,,

20	2, 7 : 6 ; 3	20
40	5, 7 : 5 ; 3	,,
...	....	...
120	17, 7 : 1 ; 3	,,
140	20, 7 : 0 ; 3	,,

and

20	6, 3 : 2 ; 7	20
40	13, 3 : 1 ; 7	,,
60	20, 3 : 0 ; 7	,, .

§ 17. For the differences 10, 20, 30, ..., if the systems are not to consist of selected terms from other systems, we must have $p\mu = 11, 21, 31, 41, 51, 61, 71, 81, 91, 101, \ldots,$. In the case of the prime numbers 11, 31, 41, 61, 71, 101, ..., p must be equal to the prime number and $\mu=1$; so that for the differences 10, 30, 40, 60, 70, 100, ..., we only have a single system and μ must $=1$. But for the difference 20 we have the three sets of values given in the preceding section; for a difference of 50, besides $p=51$, $\mu=1$, we have also $p=17$, $\mu=3$, and $p=3$, $\mu=17$; for a difference of 80, besides $p=81$, $\mu=1$, we have

* The second of these systems contains Tagliente's solution of his question in the edition of 1525 (§ 3). Instead of having only three women with 20, 40, 60 eggs, he might have had seven women with 20, 40, ..., 140 eggs, or six women, if he had excluded the case of one of the women selling the whole of the eggs at one price.

$p=9$, $\mu=9$; for a difference of 90, besides $p=91$, $\mu=1$, we have $p=13$, $\mu=7$ and $p=7$, $\mu=13$, and so on.

As examples, we may notice the following systems which correspond to differences 50 and 80, and in which the fixed-sum is taken to be equal to the least of the numbers.

50	2, 17: 16; 3	50,	80	8, 9: 8; 9	80
100	5, 17: 15; 3	,,	160	17, 9: 7; 9	,,
...	....	...	...	....	...
800	47, 17: 1; 3	,,	640	71, 9: 1; 9	,,
850	50, 17: 0; 3	,,	720	80, 9: 0; 9	,, ,

the first system containing 17 lines and the second containing 9 lines.

In all the systems if the quotients be increased or diminished by a, the numbers represented will be increased or diminished by pa, and the fixed-sum will be increased or diminished by a.

Applications to the original question, §§ 18—19.

§ 18. Returning now to the problem of women selling different numbers of eggs at the same prices and bringing back the same money, which suggested the mathematical question of the determination of a system of numbers having the same fixed-sum, it is to be noted that in general the divisor p is the number of eggs sold for a penny, that μ is the price in pence for which each of the eggs left over was sold, and that the fixed-sum is the amount of money brought back by each woman.

It follows from § 8 that it is always possible to assign to each of p women different numbers of eggs, these numbers being in arithmetical progression, such that, if sold, as far as their numbers admit, at p for a penny, and the eggs that are left over at a certain number of pence for each egg, then they all bring back the same money. In this statement it is supposed that all the eggs which a woman has may be sold at one price: but if this case is excluded the greatest number of women is $p-1$.

The system in § 9 shows that there is always at least one solution in which the number of pence brought back by each woman is the same as the least number of eggs assigned to any of the women.

§ 19. In the original problem the number of women was given, viz. 3, and also the number of eggs which each received,

viz. 10, 30, 50. The only solution, if each woman sells her eggs at both prices and as many as possible at the smaller price, is given by the system

$$\begin{array}{r|rrrr|l} 10 & 1, & 7: & 3; & 3 & 10 \\ 30 & 4, & 7: & 2; & 3 & ,, \\ 50 & 7, & 7: & 1; & 3 & ,, \end{array},$$

for if the divisor be taken to be 3, so that 3 eggs are sold for a penny and those left over at 7 pence each, the fixed-sum could not be the same for the numbers 10 and 50: but if we admit that one of the women may sell all her eggs at one price, then by taking $p=11$ and $p=21$, we have the solutions

$$\begin{array}{r|rrrr|l} 10 & 0, & 11: & 10; & 1 & 10, \\ 30 & 2, & 11: & 8; & 1 & ,, \\ 50 & 4, & 11: & 6; & 1 & ,, \end{array} \qquad \begin{array}{r|rrrr|l} 10 & 0, & 21: & 10; & 1 & 10 \\ 30 & 1, & 21: & 9; & 1 & ,, \\ 50 & 2, & 21: & 8; & 1 & ,, \end{array}.$$

These three systems can be extended so as to contain respectively 4, 6, and 11 numbers, the last lines of each being respectively

$$70\,|\,10,\ 7:\ 0;\ 3\,|\,10, \quad 110\,|\,10,\ 11:\ 0;\ 1\,|\,10, \quad 210\,|\,10,\ 21:\ 0;\ 1\,|\,10.$$

It will be noticed that the second system consists of alternate numbers selected from the complete system (in which the difference is 10):

$$\begin{array}{r|rrrr|l} 10 & 0, & 11: & 10; & 1 & 10 \\ 20 & 1, & 11: & 9; & 1 & ,, \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 100 & 9, & 11: & 1; & 1 & ,, \\ 110 & 10, & 11: & 0; & 1 & ,, \end{array}.$$

In all these systems it is supposed (i) that a woman must sell her eggs as far as her stock permits at the lower price (of so many for a penny), and that it is only those that are then left over which are to be sold at the higher price; and (ii) that a woman may sell all her eggs at one price. There is nothing in the statement of the question as originally proposed to impose the limitation (i), but it seems to have been clearly intended by the early writers. As for (ii) it is almost certain that this was not regarded as permissible, and that solutions in which all the eggs were sold at one price were excluded.

Remarks on Widman's procedure, §§ 20-21.

§ 20. I was led to consider the mathematical basis of the original problem by Widman's construction of similar questions which was quoted in § 6. After giving the original question and its solution he seems to have been tempted to examine the principles on which the solution depended and to discover how questions of the same type, in which more than three women were engaged, could be devised.

In the solution of the original problem where there are three women, the numbers in the third column (starting from the bottom) are 1, 2, 3; those in the second column are the divisor 7, and those in the first column are obtained by starting with the divisor 7 and continually subtracting the number 3. Widman forms his first system in an exactly similar manner, the divisor chosen being 9. In his second system he takes 13 as divisor and supposes that there are four women. The numbers in the third column are 1, 2, 3, 4; and he forms the first column by writing 13 at the bottom and continually subtracting 4 (the top number of the third column) to obtain the numbers above it. In his last system the divisor is 30 and the number of women five: there are 5 lines and the numbers in the first column are obtained by writing 30 at the bottom and continually subtracting 5.

Thus, using the form of representation adopted in this paper (which differs from Widman's only by adding a fourth column giving the multiplier, putting the total number of eggs to the left instead of the right, and inserting the fixed-sum on the right), Widman's general problem and solution when p is the divisor and r is the number of women is

$$\begin{array}{l|llll|l}
p^2-(r-1)pr+r & p-(r-1)r, & p: & r; & r & p+r \\
p^2-(r-2)pr+r-1 & p-(r-2)r, & p: & r-1; & r & ,, \\
\cdots\cdots\cdots & \cdots\cdots & \cdots & \cdots & \cdots & \cdots \\
p^2-pr+2 & p-r, & p: & 2; & r & ,, \\
p^2+1 & p, & p: & 1; & r & ,, \;.
\end{array}$$

This gives a problem of the kind in question and its solution if $p \geqq (r-1)r$.

§ 21. It is clear that there was no need to take the multiplier to be r, the number of women, or to take p to be the lowest number in the first column: in fact, any number μ might have been taken as the multiplier, and any number q, such

that $q > \mu(r-1)$, might have been taken as the lowest number of the first column. Thus Widman's system might have been

$pq-(r-1)p\mu+r$	$q-(r-1)\mu,$	$p:$	$r;$	μ	$q+\mu$
$pq-(r-2)p\mu+r-1$	$q-(r-2)\mu,$	$p:$	$r-1;$	μ	,,
....		....			......
$pq-p\mu+2$	$q-\mu,$	$p:$	$2;$	μ	,,
$pq+1$	$q,$	$p:$	$1;$	μ	,,

and his first example, in which the divisor is 9, might have been

12	1, 9 : 3 ; 3	10	or	12	1, 9 : 3 ; 4	13, &c.
38	4, 9 : 2 ; 3	,,		47	5, 9 : 2 ; 4	,,
64	7, 9 : 1 ; 3	,,		82	9, 9 : 1 ; 4	,, .

It is to be observed that in the above system the largest number of women is given by the largest value of r for which $q-(r-1)\mu$ is positive. This largest number is therefore $1+I\left(\frac{q}{\mu}\right)$ unless μ is a divisor of q in which case it is $I\left(\frac{q}{\mu}\right)$.

§ 22. Returning to Widman's examples it would seem that, noticing that the last line in the table of the original question was 7, 7, 1 and that the number of women was the same as the number of pence paid for an egg at the second sale, he constructed his other tables on this model, taking as his fundamental conditions (i) that the lowest line should be $p, p, 1$, where p is the number of eggs sold for a penny at the first sale, and (ii) that the number of pence for which an egg was sold at the second sale should be the same as the number of women. Subject to these limitations Widman's examples were well chosen to illustrate the different types. Thus in his first example in which the lowest line is 9, 9, 1 and the number of women is 3 the first number in the first line is not unity but another number, 3. In his second example in which the number of women is 4, the first number in the first line is 1. This first number is 1 whenever p and r are connected by the relation $p=r^2-r+1$; in this case $r=4$, so that $p=13$. In his last example where the number of women is 5 he assigns a large number 30 to p, and the first number in the first line is 10. In this example the table could

have been continued upwards one line more, this new line being 5, 30, 6, so that the first women could have had 156 eggs and sold them at 30 to the penny and at 5 pence each and brought back 35 pence the same as the other 5 women, who had 305, ..., 901 eggs.

§23. It seems singular that Widman should have founded his systems on the last line of the original system instead of on the first line 1, 7, 3, but presumably his intention was that the woman with the largest number of eggs should have 1 left over, that the next woman should have 2 left over, and so on. It was therefore more convenient to start with 1 in the third column, *i.e.* to start from the bottom.

With respect to the number of pence paid for an egg at the second sale there is no reason why this should have been taken to be the same as the number of women. It is essential that the number of eggs left over by the first woman should be equal to or greater than the number of women, but the number of pence for which each egg is sold at the second sale may be taken to be any number less than the number of women and, if p is large enough, to be greater than the number of women. It would seem that the fact that in the original question there were three women and the price of each egg was three pence led Widman to take these numbers to be the same in his examples, but he might equally well have joined by his curved lines the numbers below the first number in the first column not only to the top number in the third column but to any of the numbers below it. Thus taking the second example he might have formed the systems

4	13	4	56	,	7	13	4	95	,	10	13	4	134
7	13	3	94		9	13	3	120		11	13	3	146
10	13	2	132		11	13	2	145		12	13	2	158
13	13	1	170		13	13	1	170		13	13	1	170 ,

in which we may suppose the second, third, and fourth numbers in the first column to be joined by curved lines (as in Widman's diagram on p. 5) respectively to 3, 2, 1 in the three systems, so that at the second sale the eggs were sold for threepence, twopence, and one penny respectively, the fixed sums being 16, 15, 14. The first system could be continued upwards so as to apply to 5 women, the second to 7 women and the third to 13 women.

Widman seems to have taken no care in his examples that the numbers of eggs given to the women should show any regularity of form, but there is some regularity in the second of these systems (of the same kind as in the original system where the numbers are 10, 30, 50), the complete system being

1	13	7	20
3	13	6	45
5	13	5	70
...	...	...	...
13	13	1	170

§ 24. Presumably Widman's object* was merely to give a direct method of forming similar problems, with different prices and different numbers of women, and in this he succeeded. It is very noticeable, considering the period in which he wrote, that he should have felt sufficient interest in the original question to desire to examine the principles upon which it depended and form others like it: and his work is remarkable as an early example of an attempt to generalise a problem.

The form of the original question as given by Blasius, § 25.

§ 25. The question of the eggs occurs, but with different numbers, in Blasius's *Liber Arithmetice Practice* (Paris, 1513).† It there takes the form that three youths having 8, 17, and 26 eggs go to market at 7 in the morning with instructions from their masters that no one of them shall sell at a greater price than the others and yet that they shall all bring back the same money. They sell their eggs at 5 for a turonus and thus receive 1, 3, and 5 turoni with 3 eggs, 2 eggs, and 1 egg left over. At 11 o'clock some merchants come to buy eggs and finding very few in the market pay 2 turoni for an egg, so that each of the youths brings back 7 turoni.

The question and solution correspond to the system

8	1,	5:	3;	2	7
17	3,	5:	2;	2	,,
26	5,	5:	1;	2	,, .

* I am assuming that the additional systems are due to Widman, but he may of course have been merely following some previous writer.

† "Liber Arithmetice Practice Astrologis Phisicis et Calculatoribus admodum utilis". By Johannes Martinus Blasius (Paris, 1513). The question is 'Decimaquarta regula' on F iii.

It will be noticed that, expressed in the manner of Widman's diagrams (§§5 and 6), Blasius's lowest line is p, p, 1, and he joins the second and third numbers in the first column by curved lines to the second number (not the top number) in the third column. Blasius seems, like Widman, to have desired the first three numbers in the third line to be of the form p, p, 1; otherwise if he had taken the number of eggs sold for a turonus to be 4, the number of turoni brought back would have been the same as the least number of eggs and the common difference (see §28).

Blasius's explanation of the two prices is practically the same as Tagliente's (§3) and precedes it by two years.

Systems in which the common difference and the fixed-sum are the same, §26.

§26. In the original problem (§1) the numbers of eggs were 10, 30, 50, the common difference being 20, and the number of pence brought back 10; and in Tagliente's second problem (§3) the numbers of eggs were 20, 40, 60, the common difference being 20, and the number of pence brought back 20; so that in the latter problem the least number of eggs, the common difference, and the number of pence brought back are the same. A general system in which these three numbers are all the same was given in §9, and it is interesting to notice some of the simplest numerical systems derivable from it which possess this property.

The general system is

$$\begin{array}{r|lllll|l}
p\mu-1 & \mu-1, & p: & p-1; & \mu & & p\mu-1 \\
2(p\mu-1) & 2\mu-1, & p: & p-2; & \mu & & ,, \\
\cdots\cdots & \cdots\cdots & & & & & \cdots \\
p(p\mu-1) & p\mu-1, & p: & 0; & \mu & & ,, \;,
\end{array}$$

and if we take $\mu=p$, it becomes

$$\begin{array}{r|lllll|l}
p^2-1 & p-1, & p: & p-1; & p & & p^2-1 \\
2(p^2-1) & 2p-1, & p: & p-2; & p & & ,, \\
\cdots\cdots & \cdots\cdots & & & & & \cdots \\
p(p^2-1) & p^2-1, & p: & 0; & p & & ,, \;,
\end{array}$$

the number of numbers in both systems being p.

Taking $p=3, 4, 5, \ldots$, the second system gives

8	2, 3: 2; 3	8,	15	3, 4: 3; 4	15,	24	4, 5: 4; 5	24
16	5, 3: 1; 3	,,	30	7, 4: 2; 4	,,	48	9, 5: 3; 5	,,
24	8, 3: 0; 3	,,	45	11, 4: 1; 4	,,	...	....	...
			60	15, 4: 0; 4	,,	120	24, 5: 0; 5	,, ,

and, by taking $p=3, 4, 5$ and $\mu=2$ in the first system, we have

5	1, 3: 2; 2	5,	7	1, 4: 3; 2	7,	9	1, 5: 4; 2	9
10	3, 3: 1; 2	,,	14	3, 4: 2; 2	,,	18	3, 5: 3; 2	,,
15	5, 3: 0; 2	,,	21	5, 4: 1; 2	,,	...	....	...
			28	7, 4: 0; 2	,,	45	9, 5: 0; 2	,,

In all these systems the quotients in the first column of the table increase by the number in the last column (the multiplier) and the number of numbers in the system is equal to the number in the second column. If the last line (in which a zero occurs in the third column) is to be omitted, the number of numbers in the system is one less, and is therefore equal to the third number in the first line of the table (which is always less by one than the second number).

Regarded as a solution of an egg problem, the number in the second column is the number of eggs sold for a penny, and that in the last column is the number of pence obtained for each egg left over. The greatest number of women selling the eggs is the second or third number in the first line of the table according as it is or is not permissible for a woman to sell all her eggs at the same price.

Remarks on certain special systems, §§ 27–29.

§ 27. By putting $\mu=1$ in the general system in the preceding section we obtain the following simple system in which the smallest number, the common difference and the fixed-sum are all the same, viz.

$p-1$	$0\ ,\ p:\ p-1;\ 1$	$p-1$
$2(p-1)$	$1\ ,\ p:\ p-2;\ 1$	,,
....	....	......
$(p-1)^2$	$p-2,\ p:\ 1\ ;\ 1$	,,
$p(p-1)$	$p-1,\ p:\ 0\ ;\ 1$	,, ,

in which the numbers in the first column increase by unity and those in the third column decrease by unity.

Considered as an egg problem, the simplest case in which three women sell their eggs at different prices and bring back the same money is given by

2	0, 3: 2; 1	2
4	1, 3: 1; 1	,,
6	2, 3: 0; 1	,, ,

the woman who has 6 eggs selling them all at 3 a penny, the woman who has 4 selling 3 for a penny and the one left over for a penny, and the woman who has 2 selling both at a penny each.

If all the women are to sell their eggs at two prices, the simplest case (*i.e.* with the least number of eggs) is

7	1, 4: 3; 1	4
10	2, 4: 2; 1	,,
13	3, 4: 1; 1	,, ,

the eggs being first sold at 4 a penny and those that are left over at a penny each. If the number of pence brought back is to be the same as the smallest number of eggs, the simplest case is given by a system in the preceding section, viz.

7	1, 4: 3; 2	7
14	3, 4: 2; 2	,,
21	5, 4: 1; 2	,, ,

in which the common difference is also the same as the smallest number and the number of pence brought back. The next simplest cases under the same conditions are

9	1, 5: 4; 2	9,	11	2, 4: 3; 3	11,	11	1, 6: 5; 2	11
18	3, 5: 3; 2	,,	22	5, 4: 2; 3	,,	22	3, 6: 4; 2	,,
27	5, 5: 2; 2	,,	33	8, 4: 1; 3	,,	33	5, 6: 3; 2	,, .

§ 28. Blasius took his lowest line to be of the form p, p : 1 ; μ and if he had taken $p=4$, $\mu=1$ instead of $p=5$, $\mu=2$ he would have obtained the even simpler system

11	2, 4: 3; 1	5
14	3, 4: 2; 1	,,
17	4, 4: 1; 1	,, .

If he had taken his lowest line to be 5, 4: 1; 2 instead of 5, 5: 1; 2 he would have obtained the system given in the preceding section in which the numbers of eggs are 7, 14, 21 and the sum brought back is 7 pence.

§ 29. When the numbers of eggs are 10, 30, 50 there is but one solution of the question, if zero values are excluded, viz. that given by Widman and earlier writers: but for the numbers 20, 40, 60 there are two solutions, viz.

20	2,	7:	6;	3	20	and	20	1,	11:	9;	1	10
40	5,	7:	5;	3	„		40	3,	11:	7;	1	„
60	8,	7:	4;	3	„		60	5,	11:	5;	1	„ ,

the first of which is that given by Tagliente.

In all the preceding systems as many as possible of the eggs are sold at the smaller price and only those that are left over at the larger price, *i.e.* the numbers in the third column are always less than the number in the second column.

PART II.

PARTITIONS OF NUMBERS INTO THE FORM $p\alpha + \beta$ WHERE $\alpha + \mu\beta$ IS THE SAME.

Systems in which the first and third columns of the table are not necessarily quotients and remainders, §§ 30–36.

§ 30. In all that precedes I have generalised the original puzzle-question by dividing the numbers of the system by p, thus forming quotient and remainder, the fixed-sum being the quotient + μ times the remainder. But (except that in the original question this affords some justification for the two prices, viz. that only those which are left over were subsequently sold at the higher price) there is no reason why the numbers in the first and third columns should be restricted to quotients and remainders, *i.e.* why the numbers in the third column should be less than p; and if we remove this restriction the numbers of the system are to be partitioned into $p\alpha + \beta$, where α and β are subject to the sole condition that $\alpha + \mu\beta$ is constant.

The solutions of this extended question will now be examined. It will be seen that the transition to the more general partitionment may be effected by deriving the other systems from that already considered, in which the first and third columns are quotients and remainders.

§ 31. Supposing n to be the lowest number of a system and to be $= ap + b$, where $b < p$, then from § 13 we have

$$\begin{array}{r|l|c} n & a \quad , \; p: \; b \quad ; \; \mu & a + \mu b \\ n + p\mu - 1 & a + \mu \; , \; p: \; b - 1 \; ; \; \mu & ,, \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots \\ n + b(p\mu - 1) & a + b\mu, \; p: \; 0 \; ; \; \mu & ,, \; . \end{array}$$

This system may conveniently be called the quotient-system as the first column in the table contains the quotients when the numbers of the system are divided by p. By subtracting unity from the numbers in the first column of the table and adding p to the numbers in the third column and continuing the table downwards till 0 appears in the third column we obtain the system

$$\begin{array}{r|l|c} n & a - 1 \quad , \; p: \; b + p \quad ; \; \mu & a + \mu b + p\mu - 1 \\ n + p\mu - 1 & a - 1 + \mu \quad , \; p: \; b + p - 1 \; ; \; \mu & ,, \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\ n + (b + p)(p\mu - 1) & a - 1 + (b + p)\mu, \; p: \; 0 \; ; \; \mu & ,, \; , \end{array}$$

and by subtracting 2 from the numbers in the first column and adding $2p$ in the third column and continuing the table we obtain

$$\begin{array}{r|l|c} n & a - 2 \quad , \; p: \; b + 2p \quad ; \; \mu & a + \mu b + 2(p\mu - 1) \\ n + p\mu - 1 & a - 2 + \mu \quad , \; p: \; b + 2p - 1 \; ; \; \mu & ,, \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\ n + (b + 2p)(p\mu - 1) & a - 2 + (b + 2p)\mu, \; p: \; 0 \; ; \; \mu & ,, \; , \end{array}$$

and in general we have

$$\begin{array}{r|l|c} n & a - r \quad , \; p: \; b + rp \quad ; \; \mu & a + \mu b + r(p\mu - 1) \\ n + p\mu - 1 & a - r + \mu \quad , \; p: \; b + rp - 1 \; ; \; \mu & ,, \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots\cdots \\ n + (b + rp)(p\mu - 1) & a - r + (b + rp)\mu, \; p: \; 0 \; ; \; \mu & ,, \; , \end{array}$$

the last system of the series being

$$\begin{array}{r|l|c} n & 0 \; , \; p: \; n \quad ; \; \mu & n\mu \\ n + p\mu - 1 & \mu \; , \; p: \; n - 1 \; ; \; \mu & ,, \\ \cdots\cdots & \cdots\cdots\cdots\cdots & \cdots \\ np\mu & n\mu, \; p: \; 0 \; ; \; \mu & ,, \; . \end{array}$$

§ 32. If n is not the lowest number of the system of numbers, *i.e.* of the arithmetical progression to which it belongs, the table can be extended downwards to numbers below n. Writing the representations of the numbers less than n below the n-line, and including the n-line itself, this portion of the table for the quotient system is

n	$a \quad , \; p: \; b \quad ; \; \mu$	$a+\mu b$
$n-(p\mu-1)$	$a-\mu \; , \; p: \; b+1 \; ; \; \mu$	,,
$n-2(p\mu-1)$	$a-2\mu, \; p: \; b+2 \; ; \; \mu$	,,
....	....	..........,

the system being continued so long as the numbers in the first column remain positive. The number of lines below the n-line is therefore $I\left(\dfrac{a}{\mu}\right)$, where $I\left(\dfrac{a}{\mu}\right)$ denotes the greatest integer in $\dfrac{a}{\mu}$.* The corresponding portion of the r^{th} non-quotient system is

n	$a-r \quad , \; p: \; b+rp \quad ; \; \mu$	$a+\mu b+r(p\mu-1)$
$n-(p\mu-1)$	$a-r-\mu \; , \; p: \; b+rp+1 \; ; \; \mu$	,,
$n-2(p\mu-1)$	$a-r-2\mu, \; p: \; b+rp+2 \; ; \; \mu$	,,
....	....	.... ,

the number of lines below the n-line being $I\left(\dfrac{a-r}{\mu}\right)$.

Thus, corresponding to the given numbers n, p, μ, the numbers of lines in the different complete systems of representations are

$$1+b+I\left(\frac{a}{\mu}\right), \quad 1+b+p+I\left(\frac{a-1}{\mu}\right), \ldots$$

$$1+b+rp+I\left(\frac{a-r}{\mu}\right), \quad 1+b+ap\,(=n+1).$$

---

* When the condition that the number in the third column of the table must be $<p$ is removed, the descending branch always continues so long as the numbers in the first column are positive, and does not end when the number in the third column reaches $p-1$. Thus in the first example in § 14 the descending branch is

24	4, 5 : 4 ; 2	12
15	2, 5 : 5 ; 2	,,
6	0, 5 : 6 ; 2	,, ,

and does not end as in § 14 when the number in the third column is one less than that in the second, *i.e.* with the number 24.

§ 33. The r^{th} non-quotient system contains rp more lines in the ascending branch (*i.e.* corresponding to numbers greater than n) than the quotient system.

If q be the smallest quotient in the descending branch of the quotient system (*i.e.* the quotient corresponding to the smallest number represented in the system), q being necessarily less than μ, then a line disappears from this branch in the $(q+1)^{\text{th}}$ non-quotient system, and another line disappears in every μ^{th} system afterwards. Thus if $r = k\mu + q + 1 + \epsilon$, where $\epsilon < \mu$, then the r^{th} non-quotient system contains $1+k$ fewer lines than the quotient system; and this number, $1+k$,

$$= 1 + \frac{r-q-1-\epsilon}{\mu} = I\left(\frac{r-q+\mu-1}{\mu}\right).$$

This result may be identified with

$$I\left(\frac{a}{\mu}\right) - I\left(\frac{a-r}{\mu}\right),$$

which is the difference between the numbers of lines in the quotient system and in the r^{th} non-quotient system, for

$$q = a - I\left(\frac{a}{\mu}\right)\mu, \quad \text{whence } I\left(\frac{a}{\mu}\right) = \frac{a-q}{\mu},$$

and therefore this difference is equal to

$$\frac{a-q}{\mu} - I\left(\frac{a-r}{\mu}\right).$$

Now it can be shown that g being a multiple of μ,

$$\frac{g}{\mu} - I\left(\frac{h}{\mu}\right) = I\left(\frac{g-h+\mu-1}{\mu}\right),$$

so that the difference in question

$$= I\left(\frac{r-q+\mu-1}{\mu}\right).$$

If $p > 1$ the quotient system must contain the fewest lines of any of the systems, but if $p = 1$ two consecutive systems have the same number of lines when the descending branch loses a line.

§ 34. Starting with the quotient system, each of the other systems is derived from its predecessor by subtracting unity from the first number in the line of the table which represents

n until zero is reached, making the requisite changes in the table by which the lines below n in the ascending branch of the table (corresponding to numbers greater than n) are increased in number, and those in the descending branch (corresponding to numbers less than n) diminish in number and at length disappear.

As each system is dependent upon the first number in the line representing n it is convenient to call this number the *leading number* of the system, *i.e.* the leading number in a system is the number in the first column in the line corresponding to the given number n.

In the preceding sections the lines in the table corresponding to the ascending and descending branches (*i.e.* to the numbers represented which are greater than n and those that are less than n) have been written separately and placed below the n-line: but in future both will be included in one system, the lines representing the numbers less than n being written above the n-line, so that in the complete system the highest line will correspond to the least number represented, and the lowest line to the greatest number represented. Thus the lines above the n-line will correspond to smaller numbers than n and those below to larger numbers.

§ 35. In § 31 the other systems in the solution of the more extended question were deduced from the quotient system which had been already obtained, the last system being that in which the representation of the given number n is $|0, p:n;\mu|$; but in considering *ab initio* the general partitionment into the form $p\alpha+\beta$ where α and β are unrestricted, *i.e.* so that the general representation of n is $|\alpha, p:\beta;\ \mu|$, it is more natural to begin with the representation $|0, p:n;\ \mu|$ and to deduce the other systems by continually increasing by unity the leading number, *i.e.* the first number in the representation of n (the third number in the representation being continually decreased by p) until a is reached, where a is the quotient when n is divided by p. Thus in general the system in which the representation of n is $|\alpha, p:\beta;\ \mu|$ is the $(\alpha+1)^{\text{th}}$ system; and the values of α in the systems increase by a unit from 0 to $I\left(\frac{n}{p}\right)$, while those of β decrease by p from n to $n-pI\left(\frac{n}{p}\right)$. The number of systems is $I\left(\frac{n}{p}\right)+1$: and the number of lines in the $(\alpha+1)^{\text{th}}$ system, *i.e.* in the system which has α as the leading number is

$$I\left(\frac{\alpha}{\mu}\right)+1+\beta,$$

or, expressed wholly in terms of α, it is

$$I\left(\frac{\alpha}{\mu}\right)+1+n-p\alpha.$$

§ 36. Hitherto it has been supposed that n, p, μ were connected in some specified manner; but a more general question, which will now be considered, is to suppose that n and the common difference are given and that it is desired to find all the systems of numbers having the given difference and which contain n, all possible values of p and μ being included.

The general question of the partitioning of a series of numbers having a common difference into the form $p\alpha+\beta$, where $\alpha+\mu\beta$ is constant, §§ 37–39.

§ 37. It is supposed that a number n is given and also the common difference d and that it is required to obtain the partitions of the system of numbers ..., $n-2d$, $n-d$, n, $n+d$, $n+2d$, ..., to which n belongs, into the form $p\alpha+\beta$ where $\alpha+\mu\beta$ is the same for the same system, all possible values of p and μ being included. It is convenient to call p and μ the modulus and the multiplier.

§ 38. In order, when d and n are given, to find all the systems of numbers which have the same fixed sum, it is necessary to determine all the possible values of p, μ. To obtain these values we first find all the divisors of d: let there be 1, l_1, l_2, ..., d. Let l be any one of these divisors, and let s be any divisor of $\dfrac{d}{l}+1$ and s' its conjugate, then s, s' is a pair of values of p, μ, and for every divisor s (including unity and $\dfrac{d}{l}+1$ itself) there is such a pair of values. It is convenient to call the systems corresponding to $l=1$ *principal* systems. The values of p in these systems are the divisors of $d+1$, and the common difference of the numbers represented is d. In the systems corresponding to the divisor l of d (l not being unity), the values of p are the divisors of $\dfrac{d}{l}+1$, and the common difference is $\dfrac{d}{l}$, and from them systems in which the common difference of the numbers represented is d are derivable by selecting every l^{th} line.

With each pair of values of p, μ we start with the system of which the n-line is $|0, p : n ; \mu|$ and derive from it all the

other systems having the same values of p, μ by continually increasing the first number in the n-line by unity, *e.g.* so that the n-line in the second system is $|1, p: n-p; \mu|$ till we reach the system $|a, p: n-pa; \mu|$ where a is the quotient when n is divided by p.

§ 39. If we denote by $\phi(k)$ the number of divisors of any number k, unity and k both included, then there are $\phi(d+1)$ values of p, μ for which $l=1$ (*i.e.* which give principal systems), and for each value of l there are $\phi\left(\frac{d}{l}+1\right)$ systems, there being two values (viz. $p=1$, $\mu=2$; and $p=2$, $\mu=1$) in the case of $l=d$. Thus the total number of values of p, μ is

$$\phi(d+1)+\phi\left(\frac{d}{l_1}+1\right)+\phi\left(\frac{d}{l_2}+1\right)+\ldots+\phi(2)$$

and the number of systems corresponding to the pair of values p, μ is $I\left(\frac{n}{p}\right)+1$.

Systems in which $d=9$ and $n=33$, §§ 40–41.

§ 40. As an example take $d=9$ and $n=33$ as in § 14. The divisors of 9 are 1, 3, 9. Taking first the divisor $l=1$ which gives the principal systems we write down the divisors of $d+1=10$ which are 10, 5, 2, 1; and the values of p, μ are 10, 1; 5, 2; 2, 5; 1, 10.

For $p=10$, $\mu=1$, the initial system, *i.e.* the system in which the leading number is 0, is

33	0, 10: 33; 1	33
42	1, 10: 32; 1	,,
...	....	...
330	33, 10: 0; 1	,, ,

and by increasing the leading number (*i.e.* the first number in the line corresponding to 33) by unity, we obtain the following three systems:

24	0, 10: 24; 1	24,	15	0, 10: 15; 1	15,	6	0, 10: 6; 1	6
33	1, 10: 23; 1	,,	24	1, 10: 14; 1	,,	...	....	...
...	....	...	33	2, 10: 13; 1	,,	33	3, 10: 3; 1	,,
240	24, 10: 0; 1	,,	...	....	...	...	....	...
			150	15, 10: 0; 1	,,	60	6, 10: 0; 1	,,.

The procedure consists in increasing the leading number by unity, and every such increase brings into existence a new line (viz. those for 24, 15, 6) since μ is equal to 1. The numbers represented necessarily increase by 9 which $=d=p\mu-1$ and the fixed-sums also decrease by $p\mu-1=9$. The number of lines in a system is one more than the number in the third column of the first line, so that the numbers of lines in these four systems are 34, 25, 16, 7.

Taking now the values $p=5$, $\mu=2$ the initial system is

33	0, 5 : 33 ; 2	66
42	2, 5 : 32 ; 2	,,
...	....	...
330	66, 5 : 0 ; 2	,, ,

and, by increasing the leading number by unity, six other systems are deduced from it of which the second and last are

24	0, 5 : 24 ; 2	48,	6	0, 5 : 6 ; 2	12.
33	2, 5 : 23 ; 2	,,	...	....	...
...	....	...	33	6, 5 : 3 ; 2	,,
				....	...
240	48, 5 : 0 ; 2	,,	60	12, 5 : 0 ; 2	,,

The second system has been chosen because when the leading number is 2 (the value of μ) the line for 24 first appears, and similarly the lines for 15 and 6 first appear when it is 4 and 6.

The numbers of lines in the seven systems are 34, 29, 25, 20, 16, 11, 7.

Coming now to the values $p=2$, $\mu=5$, and writing down the first system, the system in which the line for 24 first appears, and the last system we have

33	0, 2: 33; 5	165,	24	0, 2: 24; 5	120,	6	1, 2: 4; 5	21
42	5, 2: 32; 5	,,	33	5, 2: 23; 5	,,	...	....	...
...	....	...	...	....	...	33	16, 2: 1; 5	,,
330	165, 2: 0; 5	,,	240	120, 2: 0; 5	,,	42	21, 2: 0; 5	,, .

As the leading number increases from 0 to 16 we obtain 17 systems, the lines for 24, 15, 6 first appearing when it is 5, 10, 15. The numbers of lines in the 17 systems are 34, 32, 30, 28, 26, 25, 23, 21, 19, 17, 16, 14, 12, 10, 8, 7, 5.

For $p=1$, $\mu=10$, the first system, the system in which the line for 24 first appears, and the last system are

33	0, 1: 33; 10	330,	24	0, 1: 24; 10	240,	6	3, 1: 3; 10	33
42	10, 1: 32; 10	,,	33	10, 1: 23; 10	,,	15	13, 1: 2; 10	,,
...	....	...	...	....	...	24	23, 1: 1; 10	,,
330	330, 1: 0; 10	,,	240	240, 1: 0; 10	,,	33	33, 1: 0; 10	,, .

There are 34 systems, the lines for 24, 15, 6 first appearing when the leading number is 10, 20, 30, and the numbers of lines in the 34 systems being 34, 33, 32, 31, 30, 29, 28, 27, 26, 25, 25, 24, 23, 22, 21, 20, 19, 18, 17, 16, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 7, 6, 5, 4.

§41. Taking now the divisor $l=3$ of 9, the values of p are the divisors of $\frac{9}{3}+1$, $=4$, so that the values of p are 4, 2, 1, and those of p, μ are 4, 1; 2, 2; 1, 4; and $p\mu-1=3$.

The initial systems (*i.e.* in which the leading number is 0) for these values of p and μ are

33	0, 4: 33; 1	33,	33	0, 2: 33; 2	66,	33	0, 1: 33; 4	132
36	1, 4: 32; 1	,,	36	2, 2: 32; 2	,,	36	4, 1: 32; 4	,,
...	....	...	...	....	...	...	....	...
132	33, 4: 0; 1	,,	132	66, 2: 0; 2	,,	132	132, 1: 0; 4	,, ,

from which, by selecting every third line, we have

33	0, 4: 33; 1	33,	33	0, 2: 33; 2	66,	33	0, 1: 33; 4	132
42	3, 4: 30; 1	,,	42	6, 2: 30; 2	,,	42	12, 1: 30; 4	,,
...	....	...	...	....	...	...	....	...
132	33, 4: 0; 1	,,	132	66, 2: 0; 2	,,	132	132, 1: 0; 4	,, .

Writing down as before the first system, the system in which the line for 24 first appears and the last system, we have

33	0, 4: 33; 1	33,	24	0, 4: 24; 1	24,	15	2, 4: 7; 1	9
42	3, 4: 30; 1	,,	33	3, 4: 21; 1	,,	24	5, 4: 4; 1	,,
...	....	...	...	....	...	33	8, 4: 1; 1	,,,
132	33, 4: 0; 1	,,	96	24, 4: 0; 1	,,			

the number of lines in the nine systems being 12, 10, 9, 9, 7, 6, 6, 4, 3;

33	0, 2: 33; 2	66, 24	0, 2: 24; 2	48, 15	4, 2: 7; 2	18
42	6, 2: 30; 2	,, 33	6, 2: 21; 2	,, 24	10, 2: 4; 2	,,
...	....	... ...	....	... 33	16, 2: 1; 2	,, ,
132	66, 2: 0; 2	,, 96	48, 2: 0; 2	,,		

the number of lines in the 17 systems being 12, 11, 10, 10, 9, 8, 9, 8, 7, 7, 6, 5, 6, 5, 4, 4, 3; and

33	0, 1: 33; 4	132, 24	0, 1: 24; 4	96, 15	9, 1: 6; 4	33
42	12, 1: 30; 4	,, 33	12, 1: 21; 4	,, 24	21, 1: 3; 4	,,
...	....	... ...	....	... 33	33, 1: 0; 4	,, ,
132	132, 1: 0; 4	,, 96	96, 1: 0; 4	,,		

the number of lines in the 34 systems being 12, 11, 11, 11, 10, 10, 10, 9, 9, 9, 8, 8, 9, 8, 8, 8, 7, 7, 7, 6, 6, 6, 5, 5, 6, 5, 5, 5, 4, 4, 4, 3, 3, 3.

Taking now $l=9$, the values of p are 2, 1 and therefore those of p, μ are 2, 1; 1, 2.

Writing down as before the first system, that in which the line for 24 first appears, and the last system, we have

33	0, 2: 33; 1	33, 24	0, 2: 24; 1	24, 24	7, 2: 10; 1	17
42	9, 2: 24; 1	,, 33	9, 2: 15; 1	,, 33	16, 2: 1; 1	,, ,
51	18, 2: 15; 1	,, 42	18, 2: 6; 1	,,		
60	27, 2: 6; 1	,,				

the numbers of lines in the 17 systems being 4, 4, 4, 4, 3, 3, 3, 3, 2, 3, 3, 3, 3, 2, 2, 2, 2;

33	0, 1: 33; 2	66, 24	0, 1: 24; 2	48, 24	15, 1: 9; 2	33
42	18, 1: 24; 2	,, 33	18, 1: 15; 2	,, 33	33, 1: 0; 2	,, ,
51	36, 1: 15; 2	,, 42	36, 1: 6; 2	,,		
60	54, 1: 6; 2	,,				

the numbers of lines in the 34 systems being 4, 4, 4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2, 3, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 2, 2, 2, 2.

General statements and formulæ, §§ 42–54.

§ 42. It is convenient to denote the leading number in any system by A, and the corresponding number in the third column (*i.e.* the third number in the line for n) by B, so that $n = Ap + B$; and to denote by a the largest value of A (*i.e.* a is the value of A in the last system, 0 being its value in the first system), and the corresponding value of B by b. Thus $n = ap + b$ where a is the quotient when n is divided by p and b is the remainder. The letters α, β will be used to denote the numbers in the first and third columns in any line of the system. Thus $(A, p : B ; \mu)$ always represents n and $|\alpha, p : \beta ; \mu|$ represents any number in the series of numbers in arithmetical progression to which n belongs and which are represented in the system.

The case $l = 1$, §§ 43–47.

§ 43. Consider first the case of principal systems (*i.e.* in which $l = 1$).

In every such system the numbers in the first column of the inferior lines (*i.e.* the lines below the n-line) increase by μ and those in the third column decrease by unity till zero is reached. In the superior lines (*i.e.* the lines above the n-line) the numbers in the first column decrease by μ, until a number less than μ is reached, and the numbers in the third column increase by unity. It is convenient thus to treat separately the inferior and superior lines, the former being reckoned downwards and the latter upwards from the n-line, which separates the two sets of lines: but regarding the system as a whole the first column regularly increases by μ from the top line to the bottom line and the third column increases by unity from the bottom line to the top line. It is to be noted that the inferior lines in a system represent numbers greater than n and the superior lines numbers less than n.

§ 44. In the series of systems for which p, μ are the same the leading number A increases from 0 to a which $= I\left(\dfrac{n}{p}\right)$. Thus there are $1 + I\left(\dfrac{n}{p}\right)$ systems. The corresponding values of B decrease from n to $n - ap$, $= n - I\left(\dfrac{n}{p}\right)p$, which is the value of b.

The number of inferior lines in a system is always equal to the number in the third column in the line for n. Thus

when the leading number is A the number of inferior lines is $B, = n - Ap$. The numbers of inferior lines in the systems diminish from n to $n - I\left(\frac{n}{p}\right) p$.

The first superior line comes into existence when the leading number A becomes $= \mu$, and in general the number of superior lines in the system in which the leading number is A is $I\left(\frac{A}{\mu}\right)$.

Thus the total number of lines in the system having A as the leading number is $1 + n - Ap + I\left(\frac{A}{\mu}\right)$. The numbers of lines in the systems therefore diminish from $1 + n$ to $1 + n - ap + I\left(\frac{a}{\mu}\right)$ where $a = I\left(\frac{n}{p}\right)$.

§ 45. A superior line comes into existence when A reaches the value μ, a second superior line appears when $A = 2\mu$, and so on. The smallest value of n for which a superior line exists has the representation $|\mu, p: 0; \mu|$, giving $n = p\mu = d + 1$. Thus, unless $n > d$, there can be no superior line, and unless $n > 2d + 1$ there cannot be a second superior line, and so on.

If $\mu = 1$ each system contains one superior line more than its predecessor, and as there are p inferior lines less, each system contains altogether $p - 1$ lines less than its predecessor.

§ 46. In the first system the smallest number represented is n and the largest is $n + nd$. In the system in which the leading number is A, the smallest number represented is $n - I\left(\frac{A}{\mu}\right) d, = n - I\left(\frac{Ap}{d+1}\right) d$, and the largest is $n + (n - pA) d$. Thus in the series of systems the smallest number decreases from n to $n - I\left(\frac{a}{\mu}\right) d, = n - I\left(\frac{ap}{d+1}\right) d$, where $a = I\left(\frac{n}{p}\right)$ and the largest number decreases from $n(d+1)$ to $n(d+1) - I\left(\frac{n}{p}\right) pd$.

§ 47. The fixed-sum in the first system, *i.e.* in which $A = 0$, is μn, and this number diminishes regularly by $p\mu - 1, = d$ as A increases by a unit, its final value (for $A = a$) being $\mu n - ad$, where $a = I\left(\frac{n}{p}\right)$.

The general case in which l is any divisor of d, §§ 48–54.

§ 48. Coming now to the more general case in which l is not necessarily unity, we have $p\mu = \frac{d}{l} + 1$, where l is any divisor of d. In any such system the numbers in the first column increase regularly from the top line to the bottom line by $l\mu$, and the numbers in the third column decrease by l. In the series of systems for which p, μ are the same, the leading number increases regularly by unity from $A = 0$ to $A = a$, where $a = I\left(\frac{n}{p}\right)$. The corresponding values of B decrease from $B = n$ to $B = n - ap$, $= n - I\left(\frac{n}{p}\right)p$. The number of systems is $1 + I\left(\frac{n}{p}\right)$.

The number of inferior lines in the system having A as leading number is $I\left(\frac{n - Ap}{l}\right)$. Thus the numbers of inferior lines in the systems decrease from $I\left(\frac{n}{l}\right)$ to $I\left(\frac{n-ap}{l}\right)$, where $a = I\left(\frac{n}{p}\right)$, the decrease between consecutive systems being irregular.

The first superior line comes into existence when $A = l\mu$, the second when $A = 2l\mu$, and so on; and in general the number of superior lines in the system having A as leading number is $I\left(\frac{A}{l\mu}\right) = I\left(\frac{Ap}{d+l}\right)$.

Thus the total number of lines in the system which has A as leading number is $1 + I\left(\frac{n - Ap}{l}\right) + I\left(\frac{Ap}{d+l}\right)$. The numbers of lines in the systems therefore diminish from $1 + I\left(\frac{n}{l}\right)$ to $1 + I\left(\frac{n - ap}{l}\right) + I\left(\frac{ap}{d+l}\right)$, where $a = I\left(\frac{n}{p}\right)$.

§ 49. If a superior line exists A must be at least equal to $l\mu$, and the smallest value of n which has $l\mu$ as a leading number is represented by $| l\mu, p, 0, \mu |$, so that $n = l\mu p = d + l$. Thus $d + l$ is the smallest value of n for which a superior line exists. Similarly, $2(d+l)$ is the smallest value of n for which two superior lines exist, and so on. If therefore n lies between $r(d+l) - 1$, and $(r+1)(d+l)$, there exists at least one system in which there are r superior lines.

This can be shown otherwise thus: if $n = ap + b$, where a is the quotient when n is divided by p and b is the remainder,

and if $a = rl\mu + r'$, where $r' < l\mu$, then there are r superior lines. Now $n = (rl\mu + r')p + b = rl\mu p + r'p + b = r(d+l) + r'p + b$; and the smallest and the largest values of r' are 0 and $l\mu - 1$, and the smallest and largest values of b are 0 and $p - 1$. Thus the smallest possible value of n is $r(d+l)$, and the largest possible value is $r(d+l) + (l\mu - 1)p + p - 1 = r(d+l) + d + l - 1$.

The greatest number of superior lines which occur in any series of systems (*i.e.* in systems for which p and μ are the same) is therefore $I\left(\frac{n}{d+l}\right)$, which depends upon l alone, and is therefore the same for all series of systems for which l is the same.

Although the last system contains the same number of superior lines whatever p, μ may be, the various superior lines come into existence for different values of A in the different series of systems, *i.e.* for $A = l\mu$, $A = 2l\mu$, ...

§ 50. Since $I\left(\frac{a}{l\mu}\right)$, $= I\left(\frac{n-b}{l\mu p}\right)$, $= I\left(\frac{n-b}{d+l}\right)$ is equal to the greatest number of superior lines, and it has just been shown that $I\left(\frac{n}{d+l}\right)$ is also equal to this number, it follows that we must have

$$I\left(\frac{n-b}{d+l}\right) = I\left(\frac{n}{d+l}\right).$$

This can be readily verified; for putting, as in the preceding section, $n = ap + b$, $a = rl\mu + r'$ we have

$$\frac{n}{l\mu p} = r + \frac{r'}{l\mu} + \frac{b}{l\mu p}, \quad \frac{n-b}{l\mu p} = r + \frac{r'}{l\mu},$$

where the least values of r' and b are 0, and their greatest values are $l\mu - 1$ and $p - 1$ respectively. Taking the least values, each expression $= r$, and taking the greatest values the equations become

$$\frac{n}{l\mu p} = r + 1 - \frac{1}{l\mu p}, \quad \frac{n-b}{l\mu p} = r + 1 - \frac{1}{l\mu},$$

and in each case the largest integer is r, so that always

$$I\left(\frac{n}{l\mu p}\right) = I\left(\frac{n-b}{l\mu p}\right).$$

§ 51. In the first system the smallest number represented is n and the largest is $n+I\left(\frac{n}{l}\right)d$. In the system in which the leading number is A, the smallest number represented is $n-I\left(\frac{A}{l\mu}\right)d$, $=n-I\left(\frac{Ap}{d+l}\right)d$, and the largest is $n+I\left(\frac{n-pA}{l}\right)d$. Thus the smallest number decreases from n to $n-I\left(\frac{ap}{d+l}\right)d$ and the largest number decreases from $n+I\left(\frac{n}{l}\right)d$ to $n+I\left(\frac{n-ap}{l}\right)d$, where $a=I\left(\frac{n}{p}\right)$.

§ 52. The fixed-sum in the first system, in which $A=0$, is μn, and this number diminishes regularly by $p\mu-1=\frac{d}{l}$ as A increases by unity, its final value for $A=a$ being $\mu n-a\frac{d}{l}$, where $a=I\left(\frac{n}{p}\right)$.

§ 53. It is to be remembered that in the systems corresponding to the divisor l of d, the complete systems have d/l as the common difference (and in these systems the numbers in the third column diminish by unity and those in the first column increase by μ), and that the systems in which d is the common difference are derived from them by selecting every l^{th} line, so that in these derived systems the numbers in the third column diminish by l and those in the first column increase by $l\mu$.

§ 54. It does not necessarily follow that when d and n are given it is possible for any given value of l to represent all the inferior numbers of the arithmetical progression. For if $n=r(d+l)+r'$ where $r'<d+l$ the least number represented is $n-rd=rl+r'$, and if this $=md+m'$ where $m'<d$, there are m numbers in the arithmetical progression which are not included in the system; that is to say the first $I\left(\frac{n-rd}{d}\right)$ numbers in the arithmetical progression are not expressed. Since $r=I\left(\frac{n}{d+l}\right)$ this number is also equal to

$$I\left(\frac{n}{d}\right)-I\left(\frac{n}{d+l}\right).$$

This result can also be reached otherwise, for $I\left(\frac{n}{d}\right)$ is the number of numbers below n in the arithmetical progression, and $I\left(\frac{n}{d+l}\right)$ is the number of superior lines in the last, or quotient, system.

The formula however needs to be modified when n is a multiple of d, for in that case the number of numbers less than n in the arithmetical progression is $I\left(\frac{n}{d}\right)-1$, since 0 is to be excluded from the arithmetical progression.

Applying the formula to the example in § 40 in which $n=33$, $d=9$, we have

$$\begin{aligned}
\text{for } l&=1, \quad I(\tfrac{33}{9})-I(\tfrac{33}{10})=3-3=0;\\
\text{,, } l&=3, \quad I(\tfrac{33}{9})-I(\tfrac{33}{12})=3-2=1;\\
\text{,, } l&=9, \quad I(\tfrac{33}{9})-I(\tfrac{33}{18})=3-1=2,
\end{aligned}$$

which agrees with §§ 40–41, where the least numbers represented by the last systems in the groups for $l=1$, $l=3$, $l=9$ are 6, 15, and 24 respectively.

Greatest number of superior lines for each value of l, § 55.

§ 55. It is interesting to determine separately for each value of p the greatest number of superior lines and to verify that they are the same for the same value of l.

For the divisor l of d the values of p are $1+\frac{d}{l}$ and its divisors $p_1, p_2, \ldots, 1$. Putting $p_0=1+\frac{d}{l}$, the largest divisor, let

$$n=a_0p_0+b_0=a_1p_1+b_1=a_2p_2+b_2=\ldots=n\,.\,1+0,$$

where the a's are quotients and the b's remainders, and $p_1, p_2, \ldots$ are all divisors of p_0.

All the other b's must be either equal to b_0 or less than b_0; for if b_0 is $<p_r$ then $a_0p_r=a_rp_r$ and $b_r=b_0$; and if $b_0>p_r$, then since b_r must be $<p_r$, it follows that $b_r<b_0$. For example, if b_0 is unity then all the b's are unity (excepting the one corresponding to $p=1$, which is always zero): if b_0 is 2 all the b's for p's greater than 2 are 2, and for $p=2$, if it occurs, b is unity.

The numbers of superior lines in the systems are

$$I\left(\frac{a_0p_0}{d+l}\right),\quad I\left(\frac{a_1p_1}{d+l}\right),\quad I\left(\frac{a_2p_2}{d+l}\right),\quad \ldots,\quad I\left(\frac{n}{d+l}\right).$$

The value of the first expression is $I\left(\frac{a_0}{l}\right)$ and since $a_r p_r = a_0\left(1+\frac{d}{l}\right)+b_0-b_r$ it follows that

$$I\left(\frac{a_r p_r}{d+l}\right)=I\left(\frac{a_0}{l}+\frac{b_0-b_r}{d+l}\right).$$

Now the largest value of b_0 is d/l and b_r must be $=$ or $<b_0$ so that the limiting values of the second term are 0 and $\frac{d}{l(d+l)}$; Let $a_0=kl+k'$ where $k'<l$, then the limiting values of

$$I\left(\frac{a_r p_r}{d+l}\right) \text{ are } I\left(k+\frac{k'}{l}\right) \text{ and } I\left(k+\frac{k'}{l}+\frac{d}{l(d+l)}\right).$$

The least value of k' is 0, and the largest value is $l-1$, so that the extreme limits of these quantities are $I(k)$ and $I\left(k+1-\frac{1}{d+l}\right)$, both of which are equal to k.

This number k may be expressed either as $I\left(\frac{n}{d+l}\right)$ or as $I\left(\frac{a_0}{l}\right)$ where a_0 is the quotient when n is divided by $1+\frac{d}{l}$, which is the largest value of p corresponding to the divisor l of p.

One-line systems, §§ 56—59.

§ 56. If $n-Ap=l+p'$, where p' is less than p, then the next system (which has $A+1$ as its leading number) will have no inferior line. Thus in the system which immediately precedes a system in which there is no inferior line, the leading number A is given by $A=\frac{n-l-p'}{p}=I\left(\frac{n-l}{p}\right)$, and this is therefore the last system in which there is at least one inferior line.

Similarly, the last system in which there are at least two inferior lines is that for which $A=I\left(\frac{n-2l}{p}\right)$, and so on.

If $n<d+l$ there is no superior line, and therefore for such values of n the last system which contains at least two lines is that for which $A=I\left(\frac{n-l}{p}\right)$, and the systems between this system and the final system for which $A=a=I\left(\frac{n}{p}\right)$ consist only of a single line, viz. the line representing n.

§ 57. These representations of n, from which neither a superior nor inferior line can be derived, do not afford solutions of the partition problem proposed (for there are not two numbers having the same fixed-sum); and therefore when $n < d + l$ the systems end with that for which $A = I\left(\dfrac{n-l}{p}\right)$: but although the remaining representations of n are not strictly systems at all, it is convenient to call them one-line systems and regard them as completing the series to which the true systems belong: for they have the same values of p and μ and are formed by the same rule, and the fixed-sums follow the same law of diminution.

§ 58. The largest value of n for which a one-line system can exist is represented by $\mid l\mu - 1,\ p:\ l-1;\ \mu \mid$ and is therefore $d + 2l - p - 1$. If $l =$ or $> p + 1$, this one-line system will be followed by one or more two-line systems, but if $l < p + 1$, it will be the last system of the series.

The largest possible value of l is d, in which case $p = 1$, $\mu = 2$, or $p = 2$, $\mu = 1$, so that the largest value of $d + 2l - p - 1$ is $3d - 2$. Thus the largest possible value of n for which a one-line system can exist is $3d - 2$, of which the representation is $\mid 2d-1,\ 1:\ d-1;\ 2 \mid 4d-3$. This is the only one-line system in the series of systems, for it is preceded by the system

$$\begin{array}{r|lcr|l} 3d-2 & 2d-2, & 1: & d;\ \ 2 & 4d-2, \\ 4d-2 & 4d-2, & 1: & 0;\ \ 2 & \ \ ,, \end{array}$$

and is followed by the $d - 1$ two-line systems

$$\begin{array}{r|c|r|c|l} 2d-2 & 0,\ 1:\ 2d-2;\ 2 & 4d-4, \ldots, 2d-2 & d-2,\ 1:\ d;\ 2 & 3d-2 \\ 3d-2 & 2d,\ 1:\ \ d-2;\ 2 & ,, \qquad 3d-2 & 3d-2,\ 1:\ 0;\ 2 & \ ,,\ . \end{array}$$

In the case of $l = d$, $p = 2$, $\mu = 1$, the number represented is $3d - 3$, and its one-line representation is

$$3d-3 \mid d-1, \quad 2: \quad d-1; \quad 1 \mid 2d-2,$$

which is followed by $I\{\frac{1}{2}(d-1)\}$ two-line systems.

§ 59. As examples of series of systems (having the same l, p, μ) ending with a one-line system and continuing after a one-line system with a system of two or more lines, I take $d = 9$, $l = 3$, $p = 2$, $\mu = 2$ as in § 40. Putting $n = 11$ and $n = 12$ the last three systems in the series are

$$\begin{array}{r|c|r|c|r|c|l} 11 & 3,\ 2:\ 5;\ 2 & 13, \quad 11 & 4,\ 2:\ 3;\ 2 & 10, \quad 11 & 5,\ 2:\ 1;\ 2 & 7 \\ 20 & 9,\ 2:\ 2;\ 2 & ,, \qquad 20 & 10,\ 2:\ 0;\ 2 & ,, & & \end{array}$$

and

12	4, 2: 4; 2	12, 12	5, 2: 2; 2	9,	3	0, 2: 3; 2	6
21	10, 2: 1; 2	,,			12	6, 2: 0; 2	,, .

For $n=33$, which was the value of n in § 40, there is no one-line system, as 33 is greater than $3d-2$, that is, than 25.

Systems in which $d=20$ and $n=10$ and 20, § 60.

§ 60. The numbers in the original question (§ 1) were 10, 30, 50, and in Tagliente's second question (§ 3) 20, 40, 60: and it is of interest to write down all the systems which contain the number 10 and have the difference 20, and all the systems which contain the number 20 and have the difference 20. Since the difference d is greater than, or equal to, the given number n, it is clear that 10 and 20 must always be the smallest numbers in the systems containing them: and since 10 and 20 are less than $d+1$, it follows that there will be one-line systems whenever $I\left(\frac{n}{p}\right)$ exceeds $I\left(\frac{n-l}{p}\right)$: and these one-line systems will never be followed by systems containing two or more lines.

Systems in which $d=20$ and $n=10$, §§ 61–62.

§ 61. If $d=20$, then $d+1=21$, and the values of p, μ in this case (in which $l=1$) are 21, 1; 7, 3; 3, 7; 1, 21. The values $p=21$, $\mu=1$ give the single system

10	0, 21: 10; 1	10
30	1, 21: 9; 1	,,
...	....	...
210	10, 21: 0; 1	,, ,

and for $p=7$, $\mu=3$, we have the two systems

10	0, 7: 10; 3	30,	10	1, 7: 3; 3	10
30	3, 7: 9; 3	,,	30	4, 7: 2; 3	,,
...	....	...	...	....	...
210	30, 7: 0; 3	,,	70	10, 7: 0; 3	,, .

For $p=3$, $\mu=7$, we have four systems

$$\begin{array}{r|l|l r|l|l r|l|l}
10 & 0,3:10;7 & 70,\ldots, & 10 & 2,3:4;7 & 30, & 10 & 3,3:1;7 & 10 \\
30 & 7,3:\ \ 9;7 & ,, & 30 & 9,3:3;7 & ,, & 30 & 10,3:0;7 & ,,\ . \\
\ldots & \ldots\ldots\ldots & \ldots & \ldots & \ldots\ldots\ldots & \ldots & & & \\
210 & 70,3:\ \ 0;7 & ,, & 90 & 30,3:0;7 & ,, & & &
\end{array}$$

Here the last system contains only two lines, the previous system containing five lines.

For $p=1$, $\mu=21$, there are 11 systems, viz.

$$\begin{array}{r|l|l r|l|l r|l|l}
10 & 0,1:10;21 & 210,\ldots, & 10 & 8,1:2;21 & 50, & 10 & 9,1:1;21 & 30 \\
30 & 21,1:\ \ 9;21 & ,, & 30 & 29,1:1;21 & ,, & 30 & 30,1:0;21 & ,,, \\
\ldots & \ldots\ldots\ldots & \ldots & 50 & 50,1:0;21 & ,, & & & \\
210 & 210,1:\ \ 0;21 & ,, & & & & & &
\end{array}$$

and the one-line system $10 \mid 10,\ 1:\ 0\ ;\ 21 \mid 10$.

Taking $l=2$, we have $d/l=10$, and the only values of p, μ are 11, 1 and 1, 11.

For $p=11$, $\mu=1$ there is the single system

$$\begin{array}{r|l|l}
10 & 0,\ \ 11:\ \ 10;\ \ 1 & 10 \\
30 & 2,\ \ 11:\ \ \ \ 8;\ \ 1 & ,, \\
\ldots & \ldots\ldots\ldots\ldots\ldots & \ldots \\
110 & 10,\ \ 11:\ \ \ \ 0;\ \ 1 & ,,,
\end{array}$$

and for $p=1$, $\mu=11$ there are eleven systems, viz. 7 systems containing three or more lines

$$\begin{array}{r|l|l r|l|l}
10 & 0,\ 1:\ 10;\ 11 & 110,\ \ldots, & 10 & 6,\ 1:\ 4;\ 11 & 50 \\
30 & 22,\ 1:\ \ 8;\ 11 & ,, & 30 & 28,\ 1:\ 2;\ 11 & ,, \\
\ldots & \ldots\ldots\ldots\ldots\ldots & \ldots & 50 & 50,\ 1:\ 0;\ 11 & ,,, \\
110 & 110,\ 1:\ \ 0;\ 11 & ,, & & &
\end{array}$$

followed by two two-line systems and two one-line systems, viz.

$$\begin{array}{r|l|l r|l|l}
10 & 7,\ 1:\ 3;\ 11 & 40, & 10 & 8,\ 1:\ 2;\ 11 & 30 \\
30 & 29,\ 1:\ 1;\ 11 & ,, & 30 & 30,\ 1:\ 0;\ 11 & ,,,
\end{array}$$

$$10 \mid 9,\ 1,\ 1,\ 11 \mid 20, \qquad 10 \mid 10,\ 1:\ 0;\ 1 \mid 10.$$

Taking $l=4$, we have $d/l=5$, and p, μ are 6, 1; 3, 2; 2, 3; 1, 6.

For $p=6$, $\mu=1$ there are two systems

$$\begin{array}{r|l|l} 10 & 0,\ 6:\ 10;\ 1 & 10, \\ 30 & 4,\ 6:\ \ 6;\ 1 & \text{,,} \\ 50 & 8,\ 6:\ \ 2;\ 1 & \text{,,} \end{array} \qquad \begin{array}{r|l|l} 10 & 1,\ 6:\ 4;\ 1 & 5 \\ 30 & 5,\ 6:\ 0;\ 1 & \text{,,}\,. \end{array}$$

For $p=3$, $\mu=2$ there are three systems

$$\begin{array}{r|l|l} 10 & 0,\ 3:\ 10;\ 2 & 20, \\ 30 & 8,\ 3:\ \ 6;\ 2 & \text{,,} \\ 50 & 16,\ 3:\ \ 2;\ 2 & \text{,,} \end{array} \quad \begin{array}{r|l|l} 10 & 1,\ 3:\ 7;\ 2 & 15, \\ 30 & 9,\ 3:\ 3;\ 2 & \text{,,} \end{array} \quad \begin{array}{r|l|l} 10 & 2,\ 3:\ 4;\ 2 & 10 \\ 30 & 10,\ 3:\ 0;\ 2 & \text{,,}, \end{array}$$

and a one-line system $10\,|\,3,\ 3,\ 1,\ 2\,|\,5$.

For $p=2$, $\mu=3$ there are four systems

$$\begin{array}{r|l|l} 10 & 0,\ 2:\ 10;\ 3 & 30, \\ 30 & 12,\ 2:\ \ 6;\ 3 & \text{,,} \\ 50 & 24,\ 2:\ \ 2;\ 3 & \text{,,} \end{array} \quad \begin{array}{r|l|l} 10 & 1,\ 2:\ 8;\ 3 & 25,\ \ldots, \\ 30 & 13,\ 2:\ 4;\ 3 & \text{,,} \\ 50 & 25,\ 2:\ 0;\ 3 & \text{,,} \end{array} \quad \begin{array}{r|l|l} 10 & 3,\ 2:\ 4;\ 3 & 15 \\ 30 & 15,\ 2:\ 0;\ 3 & \text{,,}, \end{array}$$

and two one-line systems $10\,|\,4,\ 2:\ 2;\ 3\,|\,10$, $10\,|\,5,\ 2:\ 0;\ 3\,|\,5$.

For $p=1$, $\mu=6$, the systems are

$$\begin{array}{r|l|l} 10 & 0,\ 1:\ 10;\ 6 & 60,\ \ldots, \\ 30 & 24,\ 1:\ \ 6;\ 6 & \text{,,} \\ 50 & 48,\ 1:\ \ 2;\ 6 & \text{,,} \end{array} \quad \begin{array}{r|l|l} 10 & 2,\ 1:\ 8;\ 6 & 50,\ \ldots, \\ 30 & 26,\ 1:\ 4;\ 6 & \text{,,} \\ 50 & 50,\ 1:\ 0;\ 6 & \text{,,} \end{array} \quad \begin{array}{r|l|l} 10 & 6,\ 1:\ 4;\ 6 & 30 \\ 30 & 30,\ 1:\ 0;\ 6 & \text{,,}, \end{array}$$

and four one-line systems $10\,|\,7,\ 1:\ 3;\ 6\,|\,25,\ \ldots,\ 10\,|\,10,\ 1:0;\ 6\,|\,10$.

Taking $l=5$, we have $d/l=4$ and p, $\mu=5$, 1; 1, 5.

For $p=5$, $\mu=1$, the systems are

$$\begin{array}{r|l|l} 10 & 0,\ 5:\ 10;\ 1 & 10, \\ 30 & 5,\ 5:\ \ 5;\ 1 & \text{,,} \\ 50 & 10,\ 5:\ \ 0;\ 1 & \text{,,} \end{array} \quad \begin{array}{r|l|l} 10 & 1,\ 5:\ 5;\ 1 & 6, \\ 30 & 6,\ 5:\ 0;\ 1 & \text{,,} \end{array} \quad 10\,|\,2,\ 5:\ 0;\ 1\,|\,2.$$

For $p=1$, $\mu=5$, they are

$$\begin{array}{r|l|l} 10 & 0,\ 1:\ 10;\ 5 & 50, \\ 30 & 25,\ 1:\ \ 5;\ 5 & \text{,,} \\ 50 & 50,\ 1:\ \ 0;\ 5 & \end{array} \quad \begin{array}{r|l|l} 10 & 1,\ 1:\ 9;\ 5 & 46,\ \ldots, \\ 30 & 26,\ 1:\ 4;\ 5 & \text{,,} \end{array} \quad \begin{array}{r|l|l} 10 & 5,\ 1:\ 5;\ 5 & 30 \\ 30 & 30,\ 1:\ 0;\ 5 & \text{,,}, \end{array}$$

and the five one-line systems $10\,|\,6,\ 1:\ 4;\ 5\,|\,26,\ \ldots,\ 10\,|\,10,\ 1:0;5\,|\,10$.

Taking $l=10$, we have $d/l=2$, and $p,\ \mu=3,\ 1;\ 1,\ 3$.

For $p=3$, $\mu=1$, there is the two-line system

$$\begin{array}{r|rrrr|l} 10 & 0, & 3: & 10; & 1 & 10 \\ 30 & 10, & 3: & 0; & 1 & ,, \end{array}$$

and there are three one-line systems

$$10\,|\,1,\ 3:\ 7;\ 1\,|\,8,\quad 10\,|\,2,\ 3:\ 4;\ 1\,|\,6,\quad 10\,|\,3,\ 3:\ 1;\ 1\,|\,4\ ;$$

and for $p=1$, $\mu=3$ there is the two-line system

$$\begin{array}{r|rrrr|l} 10 & 0, & 1: & 10; & 3 & 30 \\ 30 & 30, & 1: & 0; & 3 & ,, \end{array}$$

and ten one-line systems

$$10\,|\,1,\ 1:\ 9;\ 3\,|\,28,\ \ldots,\ 10\,|\,10,\ 1:\ 0;\ 3\,|\,10.$$

Taking $l=20$, we have $d/l=1$, and $p,\ \mu=2,\ 1;\ 1,\ 2$. From the former we obtain merely the six one-line systems $10\,|\,0,\ 2:\ 10;\ 1\,|\,10,\ \ldots,\ 10\,|\,5,\ 2:\ 0;\ 1\,|\,5$ and from the latter the eleven one-line systems $10\,|\,0,\ 1:\ 10;\ 2\,|\,20,\ \ldots,\ 10\,|\,10,\ 1:\ 0;\ 2\,|\,10$.

§ 62. In selecting the systems to be written down, for the various values of p and μ, in the preceding list I have included the last system which contains three lines, or, when there is no such system the last system containing the next greater number of lines, as the original question required that the three numbers 10, 30, 50, should be represented. This restriction to the number 3 was, however, arbitrary, and in the solution of the general mathematical question all systems which contain two or more lines should be included.

As mentioned in § 59 no one-line systems occurred in §§ 40–41, as when $d=9$ the largest possible value of n for which a one-line system can occur is 25. For $d=20$, the largest value of n for a one-line system to occur is 58.

Systems in which $d=20$ and $n=20$, § 63.

§ 63. In the second question (§ 3) in which $n=20$, but $d=20$ as before, the values of l, p, μ are the same, and therefore there is the same number of series of systems, but in general each series (*i.e.* series of systems corresponding to given values of p, μ) will contain more systems. When $l=1$

the systems are

20	0, 21: 20; 1	20
40	1, 21: 19; 1	,,
...	....	...
420	20, 21: 0; 1	,, ,

20	0, 7: 20; 3	60, ...,	20	2, 7: 6; 3	20
40	3, 7: 19; 3	,,	40	5, 7: 5; 3	,,
...	....	...	...	....	...
420	60, 7: 0; 3	,,	140	20, 7: 0; 3	,, ,

20	0, 3: 20; 7	140, ...,	20	6, 3: 2; 7	20
40	7, 3: 19; 7	,,	40	13, 3: 1; 7	,,
...	....	...	60	20, 3: 0; 7	,, ,
420	140, 3: 0; 7	,,			

20	0, 1: 20; 21	420, ...,	20	18, 1: 2; 21	60,	20	19, 1: 1; 21	40
40	21, 1: 19; 21	,,	40	39, 1: 1; 21	,,	40	40, 1: 0; 21	,, ,
...	....	...	60	60, 1: 0; 21				
420	420, 1: 0; 21	,,						

and the one-line system 20 | 20, 1: 0; 21 | 20.

Taking $l=2$, the systems are

20	0, 11: 20; 1	20,	20	1, 11: 9; 1	10
40	2, 11: 18; 1	,,	40	3, 11: 7; 1	,,
...	....	...	...	....	...
220	20, 11: 0; 1	,,	100	9, 11: 1; 1	,, ;

20	0, 1: 20; 11	220, ...,	20	16, 1: 4; 11	60 ...,	20	18, 1: 2; 11	40
40	22, 1: 18; 11	,,	40	38, 1: 2; 11	,,	40	40, 1: 0; 11	,, ,
...	....	...	60	60, 1: 0; 11	,,			
220	220, 1: 0; 11	,,						

and two one-line systems.

For $l=4$, the systems are

20	0, 6: 20; 1	20, …, 20	2, 6: 8; 1	10, 20	3, 6: 2; 1	5;
40	4, 6: 16; 1	,, 40	6, 6: 4; 1	,,		
…	…………	… 60	10, 6: 0; 1	,,		
120	20, 6: 0; 1	,,				

20	0, 3: 20; 2	40, …, 20	4, 3: 8; 2	20, 20	5, 3: 5; 2	15
40	8, 3: 16; 2	,, 40	12, 3: 4; 2	,, 40	13, 3: 1; 2	,, ,
…	……………	… 60	20, 3: 0; 2	,,		
120	40, 3: 0; 2	,,				

and one one-line system;

20	0, 2: 20; 3	60,…,20	6, 2: 8; 3	30,…, 20	8, 2: 4; 3	20
40	12, 2: 16; 3	,, 40	18, 2: 4; 3	,, 40	20, 2: 0; 3	,, ,
…	…………	… 60	30, 2: 0; 3	,,		
120	60, 2: 0; 3	,,				

and two one-line systems;

20	0, 1: 20; 6	120,…,20	12, 1: 8; 6	60,…,20	16, 1: 4; 6	40
40	24, 1: 16; 6	,, 40	36, 1: 4; 6	,, 40	40, 1: 0; 6	,, ,
…	…………	… 60	60, 1: 0: 6	,,		
120	120,1: 0; 6	,,				

and four one-line systems.

For $l=5$, the systems are

20	0, 5: 20; 1	20,…, 20	2, 5: 10; 1	12, 20	3, 5: 5; 1	8
40	5, 5: 15; 1	,, 40	7, 5: 5; 1	,, 40	8, 5: 0; 1	,, ,
…	…………	,, 60	12, 5: 0; 1	,,		
100	20, 5: 0; 1	,,				

and one one-line system;

$$\begin{array}{r|r|lr|r|lr|r|l}
20 & 0, 1: 20; 5 & 100, \ldots, & 20 & 10, 1: 10; 5 & 60, \ldots, & 20 & 15, 1: 5; 5 & 40 \\
40 & 25, 1: 15; 5 & ,, & 40 & 35, 1: \;\, 5; 5 & ,, & 40 & 40, 1: 0; 5 & ,, \; , \\
\ldots & \ldots\ldots\ldots\ldots & \ldots & 60 & 60, 1: \;\, 0; 5 & ,, & & & \\
100 & 100, 1: \;\, 0; 5 & ,, & & & & & &
\end{array}$$

and five one-line systems.

For $l = 10$,

$$\begin{array}{r|r|lr|r|l}
20 & 0, \; 3: \; 20; \; 1 & 20, \; \ldots, & 20 & 3, \; 3: \; 11; \; 1 & 14 \\
40 & 10, \; 3: \; 10; \; 1 & ,, & 40 & 13, \; 3: \;\;\, 1; \; 1 & ,, \; , \\
60 & 20, \; 3: \;\;\, 0; \; 1 & ,, & & &
\end{array}$$

and three one-line systems;

$$\begin{array}{r|r|lr|r|l}
20 & 0, \; 1: \; 20; \; 3 & 60, \; \ldots, & 20 & 10, \; 1: \; 10; \; 3 & 40 \\
40 & 30, \; 1: \; 10; \; 3 & ,, & 40 & 40, \; 1: \;\;\, 0; \; 3 & ,, \; , \\
60 & 60, \; 1: \;\;\, 0; \; 3 & ,, & & &
\end{array}$$

and ten one-line systems.

For $l = 20$,

$$\begin{array}{r|r|l}
20 & 0, \;\; 2: \;\; 20; \;\; 1 & 20 \\
40 & 20, \;\; 2: \;\;\;\; 0; \;\; 1 & ,, \; ,
\end{array}$$

and ten one-line systems;

$$\begin{array}{r|r|l}
20 & 0, \;\; 1: \;\; 20; \;\; 2 & 40 \\
40 & 40, \;\; 1: \;\;\;\; 0; \;\; 2 & ,, \; ,
\end{array}$$

and twenty one-line systems.

Results when $d = 10$ and $n = 10$ and 20, § 64.

§ 64. The following table shows, for each set of values of l, p, μ, and for $n = 10$ and $n = 20$, the number of systems containing at least three lines, the number containing at least two lines, and the total number of systems when one-line systems are included.

In the table, H_3 is the number of systems which contain three or more lines, H_2 the number which contain two or more lines, and H_1 the total number of systems when one-line

systems are included; h_2 is the number of two-line systems, and h_1 the number of one-line systems.

			n = 10					n = 20				
l	p	μ	H_3	H_2	H_1	h_2	h_1	H_3	H_2	H_1	h_2	h_1
1	21,	1	1	1	1	0,	0	1	1	1	0,	0
,,	7,	3	2	2	2	0,	0	3	3	3	0,	0
,,	3,	7	3	4	4	1,	0	7	7	7	0,	0
,,	1,	21	9	10	11	1,	1	19	20	21	1,	1
2	11,	1	1	1	1	0,	0	2	2	2	0,	0
,,	1,	11	7	9	11	2,	2	17	19	21	2,	2
4	6,	1	1	2	2	1,	0	3	3	4	0,	1
,,	3,	2	1	3	4	2,	1	5	6	7	1,	1
,,	2,	3	2	4	6	2,	2	7	9	11	2,	2
,,	1,	6	3	7	11	4,	4	13	17	21	4,	4
5	5,	1	1	2	3	1,	1	3	4	5	1,	1
,,	1,	5	1	6	11	5,	5	11	16	21	5,	5
10	3,	1	0	1	4	1,	3	1	4	7	3,	3
,,	1,	3	0	1	11	1,	10	1	11	21	10,	10
20	2,	1	0	0	6	0,	6	0	1	11	1,	10
,,	1,	2	0	0	11	0,	11	0	1	21	1,	20
Total		...	32	53	99	21,	46	93	124	184	31,	60

Thus the column headed H_2 gives the number of solutions of the mathematical question proposed in the case of $n = 10$ and 20, *i.e.* when n and some other number in the arithmetical progression containing n have the same fixed-sum; and the column headed H_3 gives the number of solutions when it is required that the three numbers 10, 30, 50, or 20, 40, 60 should always be represented.

Partitionment into the form $\alpha + \beta$ where $\lambda\alpha + \mu\beta$ is constant, §§ 65–67.

§ 65. It will be noticed that the systems in which p is unity give the partitionment of n, $n \pm d$, ... into the form $\alpha + \beta$ where $\alpha + \mu\beta$ is constant. This is a particular case of the general partitionment of these numbers into the form

$\alpha+\beta$ where $\lambda\alpha+\mu\beta$ is constant. Such partitions also afford solutions of the egg question when at both sales each egg is sold for an integral number of pence.

Thus for example, we have

10	$1\times1+9\times6$	55
30	$25\times1+5\times6$	,,
50	$49\times1+1\times6$	,,

where the first numbers in each of the two products are the two parts into which the number of eggs is partitioned and the second factors are the numbers of pence obtained for each egg, *i.e.* 1 egg at a penny and 9 at sixpence, or 25 at a penny and 5 at sixpence, or 49 at a penny and 1 at sixpence, produce 55 pence.

§ 66. In general, let $n=\alpha+\beta$ and $n'=\alpha'+\beta'$, and suppose $\lambda\alpha+\mu\beta=\lambda\alpha'+\mu\beta'$, then $\lambda(\alpha'-\alpha)=\mu(\beta-\beta')$, and therefore if λ and μ are prime to each, $\alpha'-\alpha$ must be divisible by μ and $\beta-\beta'$ by λ and we may take $\alpha'=\alpha+k\mu$, $\beta'=\beta-k\lambda$.

The following scheme shows a series of numbers in arithmetical progression which are so partitioned into the form $\alpha+\beta$ that $\lambda\alpha+\mu\beta$ is constant.

$A+B$	$A\times\lambda+B\times\mu$	$A\lambda+B\mu$
$A+B+\mu-\lambda$	$(A+\mu)\times\lambda+(B-\lambda)\times\mu$	,,
....	....	....
$A+B+k(\mu-\lambda)$	$(A+k\mu)\times\lambda+(B-k\lambda)\times\mu$	,, .

The numbers represented continue so long as $B-k\lambda$ remains positive, *i.e.* there are altogether $I\left(\frac{B}{\lambda}\right)+1$ numbers represented.

Using the notation described in § 12 and followed throughout in this paper (in which ; denotes rejection of the following number in the formation of the number, and multiplication by it in the formation of the fixed-sum) the preceding system may be written

n	$A\ ;\ \lambda:\ B\ ;\ \mu$	$\lambda A+\mu B$
$n+\mu-\lambda$	$A+\mu;\ \lambda:\ B-\lambda;\ \mu$	,,
....	....	....
$n+k(\mu-\lambda)$	$A+k\mu;\ \lambda:\ B-k\lambda;\ \mu$	,, .

§ 67. If the common difference is given, then if we suppose λ to be less than μ, we must put $\mu-\lambda$ equal to this common difference. If then d be the common difference, we take $\lambda = 1, 2, \ldots, n$ and the corresponding value of μ are $d+1, d+2, \ldots, d+n$.

For $\lambda = n$ there is but one system, and it consists of only two lines, viz.

$$\begin{array}{r|l|l} n & 0;\ n:\ n;\ n+d & n(n+d) \\ n+d & n+d;\ n:\ 0;\ n+d & \quad ,, . \end{array}$$

In the systems for which the values of λ is the same, the fixed-sums diminish by $\mu-\lambda$; that is by d.

The case of $d=20$, $n=10$, §§ 68–70.

§ 68. As an example I take the case of $d=20$, $n=10$. The values of λ are 1, 2, ..., 10, the corresponding values of μ being 21, 22, ..., 40.

For $\lambda = 1$, $\mu = 21$, the systems are

$$\begin{array}{r|l|l} 10 & 0;\ 1:\ 10;\ 21 & 210, \ldots, \\ 30 & 21;\ 1:\ 9;\ 21 & ,, \\ \ldots & \ldots\ldots\ldots\ldots & \ldots \\ 210 & 210;\ 1:\ 0;\ 21 & ,, \end{array}
\quad
\begin{array}{r|l|l} 10 & 8;\ 1:\ 2;\ 21 & 50, \\ 30 & 29;\ 1:\ 1;\ 21 & ,, \\ 50 & 50;\ 1:\ 0;\ 21 & ,, \end{array}
\quad
\begin{array}{r|l|l} 10 & 9;\ 1:\ 1;\ 21 & 30 \\ 30 & 30;\ 1:\ 0;\ 21 & ,,, \end{array}$$

and the one-line system $10 \mid 10;\ 1:\ 0;\ 21 \mid 10$.

The leading number, *i.e.* the first number in the line for n increases from 0 to n, the corresponding third number decreasing from n to 0.

In the following sets of systems, corresponding to the successive values of λ, I give the first system and the last systems containing three lines and two lines respectively and omit the one-line systems.

When the one-line systems are omitted the leading number increases from 0 to $10-\lambda$, and the third number decreases from 10 to λ.

For $\lambda = 2$, $\mu = 22$, the systems are

$$\begin{array}{r|l|l} 10 & 0;2:\ 10;\ 22 & 220, \ldots, \\ 30 & 22;\ 2:\ 8;\ 22 & ,, \\ \ldots & \ldots\ldots\ldots\ldots & \ldots \\ 220 & 110;2:\ 0;\ 22 & ,, \end{array}
\quad
\begin{array}{r|l|l} 10 & 6;2:4;22 & 100, \ldots, \\ 30 & 28;2:2;22 & ,, \\ 50 & 50;2:0;22 & ,, \end{array}
\quad
\begin{array}{r|l|l} 10 & 8;2:2;22 & 60 \\ 30 & 30;2:0;22 & ,, . \end{array}$$

The first and third columns can be divided throughout by 2, the fixed-sums being halved at the same time, and this is the simpler form in which to exhibit the results, but $\lambda=2$, $\mu=22$ are the actual values used in the construction of the systems (*i.e.* the first column is obtained by continually adding 22 and the third by continually subtracting 2) and it is convenient to retain them; and they also show the relations of this series of systems to the others.

For $\lambda=3$, $\mu=23$, the systems are

10	0; 3: 10; 23	230, ...,	10	4; 3: 6; 23	150, ...,	10	7; 3: 3; 23	90
30	23; 3: 7; 23	,,	30	27; 3: 3; 23	,,	30	30; 3: 0; 23	,, ,
50	46; 3: 4; 23	,,	50	50; 3: 0; 23				
70	69; 3: 1; 23	,,						

For $\lambda=4$, $\mu=24$, they are

10	0; 4: 10; 24	240, ...,	10	2; 4: 8; 24	200, ...,	10	6; 4: 4; 24	120
30	24; 4: 6; 24	,,	30	26; 4: 4; 24	,,	30	30; 4: 0; 24	,, ,
50	48; 4: 2; 24	,,	50	50; 4: 0; 24	,,			

The factor 4 can be divided out from the second and fourth columns, the corresponding fixed-sums being divided also by 4.

For $\lambda=5$, $\mu=25$, the systems are

10	0; 5: 10; 25	250, ...,	10	5; 5: 5; 25	150
30	25; 5: 5; 25	,,	30	30; 5: 0; 25	,, ,
50	50; 5: 0; 25	,,			

from which the factor 5 can be divided out.

For the values $\lambda=6$, $\mu=26$ to $\lambda=10$, $\mu=30$ the systems are

10	0; 6: 10; 26	260, ...,	10	4; 6: 6; 26	180
30	26; 6: 4; 26	,,	30	30; 6: 0; 26	,, ,

from which 2 can be divided out; and

10	0; 7: 10; 27	270, ...,	10	3; 7: 7; 27	210
30	27; 7: 3; 27	,,	30	30; 7: 0; 27	,, ,

10	0; 8: 10; 28	280, ...,	10	2; 8: 8; 28	240
30	28; 8: 2; 28	,,	30	30; 8: 0; 28	,, ,

10	0; 9: 10; 29	290, ...,		10	1; 9: 9; 29	270
30	29; 9: 1; 29	,,		30	30; 9: 0; 29	,, ,

10	0; 10: 10; 30	300
30	30; 10: 0; 30	,, .

The first set of systems in which $\lambda = 1$ and $\mu = 21$ is the same as the fourth set of systems in § 61.

§ 69. Regarded as solutions of the egg question in which 10, 30, and 50 eggs were sold, it thus appears that of these integral solutions there are 9 when they are sold at one penny each and at 21 pence each; 7 when they are sold at one penny and at 11 pence, 5 when they are sold at three pence and at 23 pence, three when they are sold at one penny and at sixpence, and one when they are sold at one penny and at fivepence: making a total of 25. If the cases in which all the eggs are sold at one price (*i.e.* when a zero occurs in the first or third columns) are excluded the number is 16.

The numbers of solutions of the partition question in which two-line systems are included are 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, making a total of 55.

§ 70. When a common factor f is divided out from λ and μ the system consists of selected lines from a system in which the common difference is $\dfrac{\mu - \lambda}{f}$. Thus, for example, taking the first of the systems in which $\lambda = 4$, $\mu = 24$, we obtain, by dividing out the common factor 4, the system

10	0; 1: 10; 6	60
30	24; 1: 6; 6	,,
50	48; 1: 2; 6	,, ,

which may be derived from the system

10	0; 1: 10; 6	60
15	6; 1: 9; 6	,,
20	12; 1: 8; 6	,,
...	....	...
60	60; 1: 0; 6	,,

by extracting the first, fifth, and ninth lines.

Remarks on the different modes of partitioning, § 71–72.

§ 71. Although not explicitly stated it seems to have been assumed by the early writers who have been mentioned that as many eggs as possible were to be sold at so many for a penny, and only those that were left over for several pence each; and Widman made the same supposition in the similar problems which he constructed.

Except as affording a reason for some eggs being retained after the first sale, and which were sold afterwards at a higher price, it would have been more natural to sell the eggs at both sales for an integral number of pence; and indeed in the earliest question of this kind that I know of, viz. that in Leonardo Pisano's *Liber Abbaci* (1202) the price at the first sale is a penny and at the second sale is a multiple of a penny (see § 114).

It therefore seems possible that the sale of eggs at so many for a penny may have been introduced in order that both sales might take place at the same market, and the question be therefore rendered more puzzling.

§ 72. Regarded as a question relating to the sale of eggs, we may suppose them to be sold (i) some at so many for a penny and the others at an integral number of pence, (ii) at an integral number of pence at both sales, (iii) at so many for a penny at both sales.

Regarded as a partition question the given series of numbers may be partitioned into two parts $p\alpha+\beta$ such that $\alpha+\mu\beta$ is constant, (ii) into two parts $\alpha+\beta$ such that $\lambda\alpha+\mu\beta$ is constant, (iii) into two parts $p\alpha+q\beta$ such that $\alpha+\beta$ is constant: or if α and β are in all three cases the parts into which the number is partitioned, then in (i) α must be a multiple of p and $\dfrac{\alpha}{p}+\mu\beta$ is constant; in (ii) $\lambda\alpha+\beta\mu$ is constant; and in (iii) α must be a multiple of p and β of q and $\dfrac{\alpha}{p}+\dfrac{\beta}{q}$ is constant.

In the notation used in this paper (§ 12) the representation of the n-line is

$$\begin{array}{lr|l|l|l}
\text{in (i)} & p\alpha+\beta & \alpha,\ p: & \beta;\ \mu & \alpha+\mu\beta, \\
\text{in (ii)} & \alpha+\beta & \alpha;\ \lambda: & \beta;\ \mu & \lambda\alpha+\mu\beta, \\
\text{in (iii)} & p\alpha+q\beta & \alpha,\ p: & \beta,\ q & \alpha+\beta.
\end{array}$$

Although there is no necessity to have different letters, it is convenient to use the separate pairs of letters p, μ; λ, μ; p, q in the three cases.

Case (i) has formed the subject of §§ 8–64.

PART III.

Treatment of the puzzle-question by Benedetto, Giovanni del Sodo, and Ghaligai, §§ 73–74.

§ 73. I now pass to the subsequent treatment of the question by other writers. In his *Pratica d'Arithmetica* (Florence, 1548)* Ghaligai gives the problem under the heading 'Ragione apostata'. It here takes the form of a master having 3 servants and giving 10 oranges to the first, 30 to the second, and 50 to the third, and directing them to go to market and, each selling at the same price, to bring back the same money. He states that Benedetto and Giovanni del Sodo were aware that this question was 'apostata' and did not belong to any fixed rule, and that it was merely of the kind that one discusses over the fire on winter evenings; but from his affection for Benedetto, who was a great man in Arithmetic (che fu grand' huomo in Arimetrica), and Giovanni del Sodo, who was his preceptor, he passes on the question to others. He supposes the oranges to be sold to two friends of the servants, the first friend buying them for 7 a penny and the second for 3 pence each.

Ghaligai also gives a second question in which there are two servants and one receives 10 oranges and the other 50: they are to sell at the same price, and the first is to bring back twice as much as the second. In this question they both sell at 7 a penny and have 3 oranges and 1 orange left over, which they sell at 13 pence each.

Benedetto as an arithmetician, § 74.

§ 74. Of Giovanni del Sodo, Ghaligai himself tells us that he was his preceptor, and that the cossic signs which he uses in his book, as well as portions of chapter x and the whole of chapter xiii (both relating to Algebra), were due to him. Benedetto is mentioned in a Latin poem on Florence published by Verino in 1583,† where the reference would seem to imply

* I quote from p. 66 (misprinted 64) of the second edition (1548). The original edition was published in 1521. Ghaligai died in 1536, and the 1548 edition is, I believe, merely a reprint of that of 1521, the number of leaves being the same. On the various editions see Boncompagni's *Bullettino*, vol. vii. (1874), p 486.

† "Vgolini Verini poetæ florentini de illustratione vrbis Florentiæ libri tres" (Paris, 1583), lib. ii , pp. 14', 15. The verses are

"Quisquis Arithmetricæ rationem discere, & artem
Vult, Benedicte, tuos libros, chartasque reuoluat,
Possit vt exiguis numeris comprendere arenam
Littoris, & fluctus omnes numerare marinos
Pythagorea domus Thuscam migrauit in vrbem
Hanc primum tenero ediscit sub flore iuuentus,
Innatamque putes Tyrrhenis cœlitus artem".

In the index he is described as 'Benedetto aritmetico'.

that he was one of the earliest teachers of arithmetic (with Arabic figures) in Florence. He is also mentioned by Poccianti in his *Catalogus* of 1589,* where he is described as "Benedictvs inter omnes Arithmeticos præstantissimus, cuius familia & si ignota, notissima tamen perpetuò erit ipsius virtus. Libros plurimos molitus est in Arithmetica, vt in secundo libro de illustratione Vrbis Florentinæ testatur Verinus his versibus". He then quotes Verino's verses (which have been given below in the note), and states that the time when he lived is not known ('quo autem tempore claruerit, desideratur').

Negri,† in his history of Florentine writers, says that Benedetto, his family being ruined, took the name Fioriva from his country towards the end of the fifteenth century, and was in great repute as a celebrated mathematician, especially in architecture and arithmetic, and that he left a treatise on architecture and many books on arithmetic. Negri refers to Verino and Poccianti, and also to Vossius‡ and to Moreri's 'Grand Dictionnaire'; but neither Vossius nor Moreri give any additional information.

Libri makes no mention of any writings of Benedetto or Giovanni del Sodo.§

Bini, in his history of the University of Perugia, states that in 1412 Antonio di Giovanni was appointed 'maestro di Aritmetica e di Abaco' at an honorarium of 170 florins, that in 1441 Pietro Segni da Firenze was appointed 'ad docendum Arismeticham, seu abicum, et geometriam' at a stipend of 50 florins, and that he held this office for 3 years, that in 1458 it was decided to appoint a 'maestro di Aritmetica', and that in 1469 he was required to be a master 'in arte geometriae, et ad docendum Abicum': he was to be appointed for 3 years and was to receive 50 soldi for each pupil. Among the occupants of the post are Benedetto di Antonio da Firenze, who was appointed in 1472 and held the office for 1 year only; and Benedetto di Ser Francesco da Firenze, who was appointed in 1480. In 1483 the latter went to Rome, asking that Antonio

* "Catalogvs scriptorvm florentinorvm omnis generis . . . avctore . . . Michaele Pocciantio" (Florence, 1589), p. 27.

† "Istoria degli scrittori fiorentini . . . del P. Giulio Negri Ferrarese . . . (Ferrara, 1722), p. 92. "Tanto illustrò Firenze con la sua Virtù Benedetto; che perduto il proprio Casato, fortì dalla sua Patria il cognome. Fioriva verso la fine del Secolo decimo quinto, in grande riputazione di celebre Matematico, e specialmente nell' Architettura, ed Aritmetica. . . ."

‡ Vossius's account is "Anno 1490 vel aliquartò antè. in honore erat Benedictus Florentinus: qui multum laudis meruit suis Arithmetices libros", and he quotes Verino's verses. This account occurs in ch. 51, § 10 (p. 314), of his "De universæ matheseos natura & constitutione liber" (Amsterdam, 1650), and in "liber iii, sive de mathesi", ch. 52, § 10, of his "De artium et scientiarum natura ac constitutione".

§ "Histoires des sciences mathématiques en Italie", vol. iii. (1840), p. 147.

di maestro Jacopo, also a Florentine, should be his substitute. Bini states that on the margin of the 'Annale' for 1482, referring to Benedetto, there is the note 'hic furore et amentia postea correptus in puteum Sr. Andreae Sr. Bartholomei se precipitem dedit, et ibi vitam finivit'. Bini suggests that this may be the Benedetto referred to by Negri and Moreri, and it seems not unlikely that at all events one of these Benedetto's may have been Ghaligai's Benedetto, who almost certainly is the same as the Benedetto mentioned by Verino, Poccianti, Negri, &c. Bini comments upon the number of Florentines who were instructors in the science of calculation.

Tartaglia's form of the puzzle-question, §§ 75–82.

§ 75. This problem was given in a greatly modified form by Tartaglia,‡ who also added others of the same kind. In Tartaglia's statement of the problem a citizen of Venice, who has 90 pearls, large and small, has 3 sons. He gives 10 of these pearls to his first son, telling him to go to the fair at Soligno and sell them and bring back 10 ducats. He gives his second son 30 pearls, telling him to go to the fair at Padua and to give for a ducat as many pearls as his brother did at Soligno, and to sell each of the large ones at the same price as his brother, and bring back 10 ducats. He gives 50 pearls to his third son, telling him to go to the fair at Florence and give as many for a ducat as his brothers, and sell each of the large ones at the same price as his brothers, and bring back 10 ducats. The question is how many pearls were given for a ducat, and for what price were the large ones sold.

The solution is that the first son sold 7 small pearls for one ducat and 3 large ones for 3 ducats each, bringing back 10 ducats, that then the second son sold 28 small pearls for 4 ducats and 2 large pearls for 6 ducats, and the third son 49 small pearls for 7 ducats and one large one for 3 ducats.

§ 76. The problem as given by Tartaglia differs in two essential respects from the form in which it was expressed by the earlier writers; first by the sum of money to be brought back, viz. 10 ducats, being given, and secondly by the objects

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* "Memorie istoriche della Perugina universita' degli studj e dei suoi professori . . . dal P. D. Vincenzio Bini" (Perugia, 1816).

† pp. 518—523, and 598.

‡ "La prima parte del general trattato di nvmeri, et misvre di Nicolo Tartaglia" (Venice, 1556). The questions referred to are nos. 136–139, pp. 256, 256′.

being of two different values, so that they were necessarily sold at different prices.

In the question as given by Tartaglia we must suppose that either the problem had been worked out beforehand by the father, and the distribution of the small and large pearls for each of the three sons had been arranged by him, or that when the first son returned and reported the manner in which he had sold the small and large pearls, the father saw that he could so distribute 30 and 50 small and large pearls to the other brothers that they too, selling at the same prices, could each bring back 10 ducats.

On the first supposition when the father gave the first son 7 small pearls and 3 large ones, it was not enough merely to tell him to bring back 10 ducats, for he might have sold the small pearls at a quarter of a ducat each and the large ones at $2\frac{3}{4}$ ducats each, &c., nor would it have sufficed to tell him that the large pearls were to be sold for an integral number of ducats each, for he might have sold the 7 small ones for 4 ducats and the 3 large ones at 2 ducats each. He must therefore have directed him to sell the small ones at so many for one ducat and the large ones at an integral number of ducats each which was equivalent to telling him exactly how they were to be sold.

On the second supposition we must assume that when the father found that his first son had sold the seven small pearls for 1 ducat and the three large ones at 3 ducats each, he saw that by giving his other sons 28 small pearls and 2 large ones, and 49 small pearls and one large one, they could each bring back 10 ducats, selling at the same prices.

But the simpler and better form of the question would be for the father to give the sons 10, 30, and 50 pearls, some small and some large, with directions to sell at the same prices, the price of each of the large ones being an integral number of ducats.

§ 77. It seems fairly clear that it was Tartaglia's object to make the question more precise, and to remove the anomaly of selling the same article at two different prices. But by doing so he has introduced other considerations, for a ducat is a gold coin containing 32 grossi, and a grosso contains 24 piccoli. It therefore admits of many subdivisions which can be expressed in grossi alone or in grossi and piccoli. Thus he could have given the sons 3 small pearls and 7 large ones; 25 small and 5 large; and 47 small and 3 large, and they could have sold the small ones at 4 grossi each and the large

ones at 1 ducat 12 grossi each; or he could have given them 5 small and 5 large; 27 small and 3 large; and 49 small and 1 large, and they could have sold the small ones at 5 grossi 8 piccoli each and the large ones at 1 ducat 26 grossi 16 piccoli each.

Expressed in the notation of §§ 12, 72 Tartaglia's solution (which corresponds to the solution of the egg question) and these two solutions, in which the small pearls were sold for 4 grossi and for 5 grossi 8 piccoli each, are

10	7; $\frac{1}{7}$: 3; 3	10,	10	3; $\frac{1}{8}$: 7; $1\frac{3}{8}$	10,	10	5; $\frac{1}{6}$: 5; $1\frac{5}{6}$	10
30	28; ,,: 2; ,,	,,	30	25; ,,: 5; ,,	,,	30	27; ,,: 3; ,,	,,
50	49; ,,: 1; ,,	,,	50	47; ,,: 3; ,,	,,	50	49; ,,: 1; ,,	,, .

§ 78. Tartaglia's question was evidently suggested by the egg question and it is due to this origin that the small pearls were sold at so many to the ducat. In the case of the eggs the sale of so many to the soldo caused some to be left over, which could be treated differently. Thus the selling of the eggs at so many to the soldo afforded some justification for the two prices; but in the case of the pearls the natural course would have been to sell both the small pearls and the large pearls at so much each, the prices being expressed in ducats, grossi, and piccoli.

§ 79. Tartaglia also gave two other questions formed on the same model. In the first of these questions a father has 99 pearls, small and large, and gives to his three sons 11, 33, 55 respectively. He bids them go to the fair and sell, each of them, the small pearls at as many to the ducat as his brothers, and the large ones at as much as his brothers, and each to bring back 11 ducats. It is required to find how many small pearls did they give for a ducat and for how much did they sell one of the large ones.

The second question is exactly similar: the father gives 16, 48, 80 pearls to his sons, and selling as before they are to bring back 16 ducats.

It will be noticed that in these two questions the three sons are merely told to go to the fair, so that presumably they go together, and arrange their prices.

§ 80. Tartaglia's solution of the first of these questions is that the small pearls were sold at 6 to the ducat, and the large ones at 2 ducats each. Thus the three sons had respectively 6, 30, 54 small pearls and obtained 1, 5, 9 ducats for them, and they had 5, 3, 1 large pearls and obtained 10, 6, 2 ducats for them. But the small pearls might have been sold at 4 grossi each and the large ones at a ducat and a half each, the distribution of small and large pearls among the sons then being 4 small and 7 large; 28 small and 5 large; 52 small and 3 large: or the small ones might have been sold at 5 grossi 8 piccoli each and the large ones at 1 ducat 2 grossi 16 piccoli each, the distribution then being 1 small and 10 large; 27 small and 6 large; and 53 small and 2 large.

Expressed, as before, in the notation of §§ 12, 72 the three solutions are

11	$6;\ \frac{1}{6}:\ 5;\ 2$	11, 11	$4;\ \frac{1}{8}:\ 7;\ 1\frac{1}{2}$	11, 11	$1;\ \frac{1}{6}:\ 10;\ 1\frac{1}{12}$	11
33	30; ,,: 3; ,,	,, 33	28; ,,: 5; ,,	,, 33	27; ,,: 6; ,,	,,
55	54; ,,: 1; ,,	,, 55	52; ,,: 3; ,,	,, 55	53; ,,: 2; ,,	,, .

§ 81. Tartaglia's solution of the second of these questions is that the small pearls were sold at 11 to the ducat and the large pearls at 3 ducats each, the numbers of small and large pearls given to the sons being 11 and 5; 44 and 4; 77 and 3.

But there are other solutions in which the small and large pearls are each sold at a price expressed in ducats, grossi, and piccoli. The systems which represent these solutions are:

16	$2;\ \frac{1}{8}:\ 14;\ 1\frac{1}{8}$	16, 16	$9;\ \frac{1}{8}:\ 7;\ 2\frac{1}{8}$	16
48	38; ,,: 10; ,,	,, 48	43; ,,: 5; ,,	,,
80	74; ,,: 6; ,,	,, 80	77; ,,: 3; ,,	,,

Since $\frac{1}{8}$th of a ducat is 4 grossi, the prices of the small and large pearls in these two solutions are 4 grossi and 1 ducat 4 grossi; and 4 grossi and 2 ducats 4 grossi:

16	$1;\ \frac{1}{16}:\ 15;\ 1\frac{1}{16}$	16, 16	$3;\ \frac{3}{16}:\ 13;\ 1\frac{3}{16}$	16
48	35; ,,: 13; ,,	,, 48	41; ,,: 7; ,,	,,
80	69; ,,: 11; ,,	,, 80	79; ,,: 1; ,,	,, .

Since $\frac{1}{16}$th of a ducat is 2 grossi, the prices of the small and large pearls in these two solutions are 2 grossi and 1 ducat

2 grossi; and 6 grossi and 1 ducat 6 grossi:

$$\begin{array}{r|llll|l} 16 & 6; & \frac{1}{6}: & 10; & 1\frac{1}{2} & 16, \\ 48 & 42; & \text{,,}: & 6; & \text{,,} & \text{,,} \\ 80 & 78; & \text{,,}: & 2; & \text{,,} & \text{,,} \end{array} \qquad \begin{array}{r|llll|l} 16 & 11; & \frac{1}{6}: & 5; & 2\frac{5}{6} & 16 \\ 48 & 45; & \text{,,}: & 3; & \text{,,} & \text{,,} \\ 80 & 79; & \text{,,}: & 1; & \text{,,} & \text{,,}\,. \end{array}$$

Since $\frac{1}{6}^{\text{th}}$ of a ducat is 5 grossi 8 piccoli the prices of the small and large pearls in these two solutions are 5 grossi 8 piccolo and a ducat and a half; and 5 grossi 8 piccoli and 2 ducats 26 grossi 16 piccoli.

The first of these two solutions might have been admitted by Tartaglia as the small pearls are sold at 6 to the ducat, and the large ones at a ducat and a half each; and he might even have admitted the solution

$$\begin{array}{r|llll|l} 16 & 7; & \frac{1}{7}: & 9; & 1\frac{2}{3} & 16 \\ 48 & 42; & \text{,,}: & 6; & \text{,,} & \text{,,} \\ 80 & 77; & \text{,,}: & 3; & \text{,,} & \text{,,}\,, \end{array}$$

in which the small pearls were sold at 7 to the ducat, and the large pearls at $1\frac{2}{3}$ ducat each (*i.e.* at 1 ducat 21 grossi 8 piccoli each).

§ 82. Solutions such as Tartaglia's, in which the small pearls are sold for so many to the ducat and an integral number of ducats is therefore obtained for them, are of arithmetical interest in themselves, as not only each part of the partition is integral but also the product of each part by its modulus is integral: but selected solutions in which the price of each small or large pearl can be expressed by means of grossi and piccoli possess no such interest as they depend upon the arbitrary manner in which the ducat has been subdivided into inferior coins. As a ducat contains 32 grossi and a grosso 24 piccoli, we obtain a solution of this kind, in which the prices can be expressed by grossi and piccoli, whenever the first modulus is a power of 2 not exceeding 2^8 or any such power multiplied by 3.

The amounts obtained for the small pearls and for the large pearls, if sold to different persons, were necessarily integral numbers of ducats or at all events expressible in ducats, grossi, and piccoli; but there is no occasion for these restrictions if each brother sold all his pearls to a single person, and in this case any values would be admissible in the second and fourth columns of the systems.

I now proceed to consider the general mathematical question of the partition of the numbers a, $3a$, $5a$, ... into two parts such that the sum of two parts, when each is multiplied by its modulus, is the equal to a.

General formulæ for the partitionment of a, $3a$, $5a$, ... into two parts such that the final sum is a, §§ 83–89.

§ 83. The object of the following investigation is to give rules for obtaining all the modes of partitioning the series of numbers a, $3a$, $5a$, $7a$, ... into two parts such that the sum of the products of one part by a modulus and of the other part by another modulus is the same for all and is equal to a, the initial number a of the system.

If x, y; x_1, y_1; ... be the parts and ρ, σ the two moduli, we have

$$\begin{aligned} x + y &= a, & \rho x + \sigma y &= a, \\ x_1 + y_1 &= 3a, & \rho x_1 + \sigma y_1 &= a, \\ x_2 + y_2 &= 5a, & \rho x_2 + \sigma y_2 &= a, \\ &\cdots\cdots & &\cdots\cdots \end{aligned}$$

which give

$$y = \frac{a(1-\rho)}{\sigma-\rho}, \quad y_1 = \frac{a(1-3\rho)}{\sigma-\rho}, \quad y_2 = \frac{a(1-5\rho)}{\sigma-\rho}, \; \ldots, .$$

Now let $\rho = \dfrac{l}{p}$, in which l is prime to p; then

$$y = \frac{a(p-l)}{p\sigma-l}, \quad y_1 = \frac{a(p-3l)}{p\sigma-l}, \quad y_2 = \frac{a(p-5l)}{p\sigma-l},$$

and we therefore have the system

$$\begin{array}{c|c|c} a & a - \dfrac{a(p-l)}{p\sigma-l}\,; \;\; \dfrac{l}{p} : \;\; \dfrac{a(p-l)}{p\sigma-l}\,; \;\; \sigma & a \\ 3a & 3a - \dfrac{a(p-3l)}{p\sigma-l}\,; \;\; ,, : \;\; \dfrac{a(p-3l)}{p\sigma-l}\,; \;\; ,, & ,, \\ 5a & 5a - \dfrac{a(p-5l)}{p\sigma-l}\,; \;\; ,, : \;\; \dfrac{a(p-5l)}{p\sigma-l}\,; \;\; ,, & ,, \\ \ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \ldots\,. \end{array}$$

Now all the numbers in the first and third columns must be integral. If we put $\dfrac{a}{p\sigma - l} = \dfrac{g}{2}$, then $\sigma = \dfrac{2a+gl}{gp}$, and the system becomes

$$\begin{array}{c|cccc|c} a & a-\dfrac{g(p-l)}{2}; & \dfrac{l}{p}: & \dfrac{g(p-l)}{2}; & \dfrac{2a+gl}{gp} & a \\ 3a & 3a-\dfrac{g(p-3l)}{2}; & ,,: & \dfrac{g(p-3l)}{2}; & ,, & ,, \\ 5a & 5a-\dfrac{g(p-5l)}{2}; & ,,: & \dfrac{g(p-5l)}{2}; & ,, & ,, \\ \dots & \dots\dots\dots & & & & \dots, \end{array}$$

in which g may be any even number, and also any uneven number if p and l are both even or both uneven.

If in the first modulus we replace l and p by gl and gp, we see that in this system p and l never occur except in the form gl and gp, and therefore all the systems obtained by giving g values other than unity are included among the systems obtained by putting $g=1$ and giving p and l all possible values in the system

$$\begin{array}{c|cccc|c} a & a-\dfrac{p-l}{2}; & \dfrac{l}{p}: & \dfrac{p-l}{2}; & \dfrac{2a+l}{p} & a \\ 3a & 3a-\dfrac{p-3l}{2}; & ,,: & \dfrac{p-3l}{2}; & ,, & ,, \quad \dots\dots\dots(\text{i}) \\ 5a & 5a-\dfrac{p-5l}{2}; & ,,: & \dfrac{p-5l}{2}; & ,, & ,, \\ \dots & \dots\dots\dots & & & & \dots. \end{array}$$

Since $p-l$ must be divisible by 2 it follows that p and l must be both even or both uneven.

This system (i) gives all the solutions of the question by assigning to l and p all the admissible values.

§ 84. In this general system $\dfrac{l}{p}$ and $\dfrac{2a+l}{p}$ are the two moduli, and p must be less than $2a+l$ as appears from the first number in the representation of a, viz. $a-\frac{1}{2}(p-l)$, which must be positive so that $p<2a+l$. Also, if there are to be at least three lines in the system, p must be greater than $5l$ (as appears from the number in the third line and third column). Thus p may have any value between $5l$ and $2a+l$, and since

p and l must be both even or both uneven, the values of p are $5l+2$, $5l+4$, ..., $2a+l-2$; and in order that a value of p may exist, $5l+2$ must be $=$ or $<2a+l-2$, *i.e.* $4l=$ or $<2a-4$; and therefore the largest possible value of l is $I(\frac{1}{2}a)-1$.

In order therefore to obtain all the systems we give to l successively the values 1, 2, 3, ..., $I(\frac{1}{2}a-1)$, and assign to p, for each value of l, the values of the uneven numbers between $5l$ and $2a+l$ if l is uneven, and the values of the even numbers between the same limits if l is even.

Thus for $l=1$, the values of p are $7,\ 9,\ \ldots,\ 2a-1$;

,, $l=2$, ,, ,, $12,\ 14,\ \ldots,\ 2a$;

,, $l=3$, ,, ,, $17,\ 19,\ \ldots,\ 2a+1$;

,, $l=4$, ,, ,, $22,\ 24,\ \ldots,\ 2a+2$;

,, $l=5$, ,, ,, $27,\ 29,\ \ldots,\ 2a+3$;

and therefore for $l=1$, the values of the moduli $\dfrac{l}{p}$ and $\dfrac{2a+l}{p}$ are

$$\frac{1}{7},\ \frac{1}{9},\ \ldots,\ \frac{1}{2a-1},\ \text{and}\ \frac{2a+1}{9},\ \frac{2a+1}{11},\ \ldots,\ \frac{2a+1}{2a-1};$$

for $l=2$, they are

$$\frac{2}{12},\ \frac{2}{14},\ \ldots,\ \frac{2}{2a},\ \text{and}\ \frac{2a+2}{12},\ \frac{2a+2}{14},\ \ldots,\ \frac{2a+2}{2a};$$

that is

$$\frac{1}{6},\ \frac{1}{7},\ \ldots,\ \frac{1}{a},\ \text{and}\ \frac{a+1}{6},\ \frac{a+1}{7},\ \ldots,\ \frac{a+1}{a};$$

for $l=3$, they are

$$\frac{3}{17},\ \frac{3}{19},\ \ldots\ \frac{3}{2a+1},\ \text{and}\ \frac{2a+3}{17},\ \frac{2a+3}{19},\ \ldots,\ \frac{2a+3}{2a+1};$$

for $l=4$, they are

$$\frac{4}{22},\ \frac{4}{24},\ \ldots\ \frac{4}{2a+2},\ \text{and}\ \frac{2a+4}{22},\ \frac{2a+4}{24},\ \ldots,\ \frac{2a+4}{2a+2};$$

that is

$$\frac{2}{11},\ \frac{2}{12},\ \ldots\ \frac{2}{a+1},\ \text{and}\ \frac{a+2}{11},\ \frac{a+2}{12},\ \ldots,\ \frac{a+2}{a+1};$$

and so on.

§ 85. The systems derived from even values of l, in which case p is also even, are all included in the system

$$\begin{array}{r|llcll|l} a & a-(p-l)\ ; & \dfrac{l}{p} : & p-l\ ; & \dfrac{a+l}{p} & & a \\ 3a & 3a-(p-3l); & \text{,,} : & p-3l; & \text{,,} & & \text{,,} \\ 5a & 5a-(p-5l); & \text{,,} : & p-5l; & \text{,,} & & \text{,,} \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots & & & & & \cdots, \end{array}$$

in which p must lie between $5l$ and $a+l$, and l has the values $1, 2, \ldots, I\{\frac{1}{4}(a-1)\}$. It is, however, more convenient to derive all the systems directly from the general system (i) of § 83 by giving to l and p all the values mentioned in § 84. It will be observed that the systems obtained from the even numbers of l in (i) and which belong to the general system just written may be derived from the system

$$\begin{array}{r|llcll|l} a & a-(p-l)\ ; & \dfrac{l}{p} : & p-l\ ; & \dfrac{a+l}{p} & & a \\ 2a & 2a-(p-2l); & \text{,,} : & p-2l; & \text{,,} & & \text{,,} \\ 3a & 3a-(p-3l); & \text{,,} : & p-3l; & \text{,,} & & \text{,,} \\ \cdots & \cdots\cdots\cdots\cdots\cdots\cdots & & & & & \cdots \end{array}$$

by selecting alternate lines.

§ 86. Taking as an example $a=10$, and applying the procedure of §§ 84 and 85 to the general formula (i) of § 83, we see that

for $l=1$, p has the values 7, 9, 11, ..., 19,
,, $l=2$,, ,, 12, 14, 16, 18, 20,
,, $l=3$,, ,, 17, 19, 21,
,, $l=4$,, ,, 22.

Thus the systems are, for $l=1$,

$$\begin{array}{r|l|l|l|l|l|l|l|} 10 & 7; \frac{1}{7} : 3; \frac{21}{7} & 10, & 6; \frac{1}{9} : 4; \frac{21}{9} & , & 5; \frac{1}{11} : 5; \frac{21}{11} & , & 4; \frac{1}{13} : 6; \frac{21}{13} \\ 30 & 28; \text{,,} : 2; \text{,,} & \text{,,} & 27; \text{,,} : 3; \text{,,} & & 26; \text{,,} : 4: \text{,,} & & 25; \text{,,} : 5; \text{,,} \\ 50 & 49; \text{,,} : 1; \text{,,} & \text{,,} & 48; \text{,,} : 2; \text{,,} & & 47; \text{,,} : 3; \text{,,} & & 46; \text{,,} : 4; \text{,,} \end{array},$$

$$\begin{array}{r|l|l|l|l|l|} 10 & 3; \frac{1}{15} : 7; \frac{21}{15} & 10, & 2; \frac{1}{17} : 8; \frac{21}{17} & , & 1; \frac{1}{19} : 9; \frac{21}{19} \\ 30 & 24; \text{,,} : 6; \text{,,} & \text{,,} & 23; \text{,,} : 7; \text{,,} & & 22; \text{,,} : 8; \text{,,} \\ 50 & 45; \text{,,} : 5; \text{,,} & \text{,,} & 44; \text{,,} : 6; \text{,,} & & 43; \text{,,} : 7; \text{,,} \end{array};$$

for $l=2$,

$$\begin{array}{r|l|l|l|l|l|}
10 & 5;\ \frac{2}{12}:5;\ \frac{22}{12} & 10, & 4;\ \frac{2}{14}:6;\ \frac{22}{14} & , & 3;\ \frac{2}{16}:7;\ \frac{22}{16} \\
30 & 27;\ ,,\ :3;\ ,, & ,, & 26;\ ,,\ :4;\ ,, & & 25;\ ,,\ :5;\ ,, \\
50 & 49;\ ,,\ :1;\ ,, & ,, & 48;\ ,,\ :2;\ ,, & & 47;\ ,,\ :3;\ ,,
\end{array},$$

$$\begin{array}{r|l|l|l|}
10 & 2;\ \frac{2}{18}:8;\ \frac{22}{18} & 10, & 1;\ \frac{2}{20}:9;\ \frac{22}{20} \\
30 & 24;\ ,,\ :6;\ ,, & ,, & 23;\ ,,\ :7;\ ,, \\
50 & 46;\ ,,\ :4;\ ,, & ,, & 45;\ ,,\ :5;\ ,,
\end{array};$$

for $l=3$,

$$\begin{array}{r|l|l|l|l|l|}
10 & 3;\ \frac{3}{17}:7;\ \frac{23}{17} & 10, & 2;\ \frac{3}{19}:8;\ \frac{23}{19} & , & 1;\ \frac{3}{21}:9;\ \frac{23}{21} \\
30 & 26;\ ,,\ :4;\ ,, & ,, & 25;\ ,,\ :5;\ ,, & & 24;\ ,,\ :6;\ ,, \\
50 & 49;\ ,,\ :1;\ ,, & ,, & 48;\ ,,\ :2;\ ,, & & 47;\ ,,\ :3;\ ,,
\end{array};$$

and for $l=4$,

$$\begin{array}{r|l|l}
10 & 1;\ \frac{4}{22}:9;\ \frac{24}{22} & 10 \\
30 & 25;\ ,,\ :5;\ ,, & ,, \\
50 & 49;\ ,,\ :1;\ ,, & ,,
\end{array}.$$

Tartaglia's solution corresponds to the first system for $l=1$, and the solutions in which the prices of the pearls are expressible in ducats, grossi, and piccoli correspond to the first and third systems for $l=2$.

§ 87. For each value of l all the other systems may be derived from the first system by diminishing the numbers in the first column by unity, increasing those in the third column by unity, and increasing the denominators of the moduli by 2.

In any given system the numbers in the first column increase by the numerator of the second modulus, and those in the third column diminish by the numerator of the first modulus, it being understood that a common factor is not to be divided out from the numerator and denominator of a modulus, *i.e.* the numbers increase by $2a+l$ and diminish by l.

§ 88. For every value of l the first and third numbers in the third line of the first system are $5a-1$ and 1, and the first and third numbers in the first line of the last system are 1 and $a-1$, and in the last system the numerator of the second modulus exceeds the denominator by 2.

Thus for $l=1$, the first and last systems are

$$\begin{array}{r|lcl|l|rcl|l}
a & a-3;\ \dfrac{1}{7}: & 3; & \dfrac{2a+1}{7} & a, & 1;\ \dfrac{1}{2a-1}: & a-1; & \dfrac{2a+1}{2a-1} & a \\[2ex]
3a & 3a-2;\ ,,: & 2; & ,, & ,, & 2a+2;\ \ ,,: & a-2; & ,, & ,, \\[2ex]
5a & 5a-1;\ ,,: & 1; & ,, & ,, & 4a+3;\ \ ,,: & a-3; & ,, & ,,,
\end{array}$$

for $l=2$, they are

$$\begin{array}{r|lcl|l|rcl|l}
a & a-5;\ \dfrac{2}{12}: & 5; & \dfrac{2a+2}{12} & a, & 1;\ \dfrac{2}{2a}: & a-1; & \dfrac{2a+2}{2a} & a \\[2ex]
3a & 3a-3;\ ,,: & 3; & ,, & ,, & 2a+3;\ \ ,,: & a-3; & ,, & ,, \\[2ex]
5a & 5a-1;\ ,,: & 1; & ,, & ,, & 4a+5;\ \ ,,: & a-5; & ,, & ,,,
\end{array}$$

and so on.

§ 89. If a is even, the largest value of l is $\frac{1}{2}a-1$, and there is but one system corresponding to this value of l, viz.

$$\begin{array}{r|rcrl|l}
a & 1; & \dfrac{l}{5l+2}: & 2l+1; & \dfrac{5l+4}{5l+2} & a \\[2ex]
3a & 5l+5; & ,,\ : & l+1; & ,, & ,, \\[2ex]
5a & 10l+9; & ,,\ : & 1; & ,, & ,,,
\end{array}$$

where $a=2l+2$.

If a is uneven, the largest value of l is $\frac{1}{2}(a-3)$ and for this value of n there are the two systems

$$\begin{array}{r|rcrl|l|rcrl|l}
a & 2; & \dfrac{l}{5l+2}: & 2l+1; & \dfrac{5l+6}{5l+2} & a, & 1; & \dfrac{l}{5l+4}: & 2l+2; & \dfrac{5l+6}{5l+4} & a \\[2ex]
3a & 5l+8; & ,,\ : & l+1; & ,, & ,, & 5l+7; & ,,\ : & l+2; & ,, & ,, \\[2ex]
5a & 10l+14; & ,,\ : & 1; & ,, & ,, & 10l+13; & ,,\ : & 2; & ,, & ,,,
\end{array}$$

where $a=2l+3$.

Derivation of systems for a, $3a$, $5a$, ... from those for $a-1$, $3(a-1)$, $5(a-1)$, ..., §§ 90—92.

§ 90. The systems of partitions for a, $3a$, $5a$, ... may be derived from those for $a-1$, $3(a-1)$, $5(a-1)$, ... by adding 1, 3, 5, ... to the numbers in the first column, and by increasing the numerator of the second modulus by 2, all else

in the system remaining unaltered. To the series of systems belonging to any value of l a new final system is to be added (having unity as the leading number) which may be derived from the preceding system in the usual manner, viz. by subtracting unity from the numbers in the first column, adding unity to those in the third column, and adding 2 to the denominators of the moduli.

Also, if a is even, a new value of l, viz. $\frac{1}{2}a-1$, occurs which gives a single new system.

Deriving in this manner the systems for 11, 33, 55 from those for 10, 30, 50 (§ 86) and writing down only the first line of each system, we obtain the following systems:

$$[11,\ 33,\ 55]$$

$(8;\ \frac{1}{7}:3;\ \frac{23}{7})$, $(7;\ \frac{1}{9}:4;\ \frac{23}{9})$, $(6;\ \frac{1}{11}:5;\ \frac{23}{11})$, $(5;\ \frac{1}{13}:6;\ \frac{23}{13})$, $(4;\ \frac{1}{15}:7;\ \frac{23}{15})$, $(3;\ \frac{1}{17}:8;\ \frac{23}{17})$, $(2;\ \frac{1}{19}:9;\ \frac{23}{19})$, $(1;\ \frac{1}{21}:10;\ \frac{23}{21})$;

$(6;\ \frac{2}{12}:5;\ \frac{24}{12})$, $(5;\ \frac{2}{14}:6;\ \frac{24}{14})$, $(4;\ \frac{2}{16}:7;\ \frac{24}{16})$, $(3;\ \frac{2}{18}:8;\ \frac{24}{18})$, $(2;\ \frac{2}{20}:9;\ \frac{24}{20})$, $(1;\ \frac{2}{22}:10;\ \frac{24}{22})$;

$(4;\ \frac{3}{17}:7;\ \frac{25}{17})$, $(3;\ \frac{3}{19}:8;\ \frac{25}{19})$, $(2;\ \frac{3}{21}:9;\ \frac{25}{21})$, $(1;\ \frac{3}{23}:10;\ \frac{25}{23})$;

$(2;\ \frac{4}{22}:9;\ \frac{26}{22})$, $(1;\ \frac{4}{24}:10;\ \frac{26}{24})$.

The systems may be completed at sight by continually adding to the first number the numerator of the second modulus and subtracting from the third number the numerator of the first modulus.

Thus the second system for $l=2$ is

$$\begin{array}{r|llll|l} 11 & 5; & \frac{2}{14}: & 6; & \frac{24}{14} & 11 \\ 33 & 29; & ,,: & 4; & ,, & ,, \\ 55 & 53; & ,,: & 2; & ,, & ,, \end{array}\ .$$

Tartaglia's solution corresponds to the first system for $l=2$, and the other two given in § 80 correspond to the third system for $l=2$ and the second for $l=4$.

§ 91. Similarly, deriving the systems for 12, 36, 60, ... from those for 11, 33, 55, ..., and writing down only the first line of each system, we find

$$[12,\ 36,\ 60]$$

$(9;\ \frac{1}{7}:3;\ \frac{25}{7})$, $(8;\ \frac{1}{9}:4;\ \frac{25}{9})$, $(7;\ \frac{1}{11}:5;\ \frac{25}{11})$, ..., $(1;\ \frac{1}{23}:11;\ \frac{25}{23})$;

$(7;\ \frac{2}{12}:5;\ \frac{26}{12})$, $(6;\ \frac{2}{14}:6;\ \frac{26}{14})$, $(5;\ \frac{2}{16}:7;\ \frac{26}{16})$, ..., $(1;\ \frac{2}{24}:11;\ \frac{26}{24})$;

$(5; \frac{3}{17} : 7; \frac{27}{17})$, $(4; \frac{3}{19} : 8; \frac{27}{19})$, $(3; \frac{3}{21} : 9; \frac{27}{21})$, ..., $(1; \frac{3}{25} : 11; \frac{27}{25})$;
$(3; \frac{4}{22} : 9; \frac{28}{22})$, $(2; \frac{4}{24} : 10; \frac{28}{24})$, $(1; \frac{4}{26} : 11; \frac{28}{26})$;
$(1; \frac{5}{27} : 11; \frac{29}{27})$.

The last system is a new one, for the value 5 for l first occurs when a is 12.

It will be noticed that there is no solution of the Tartaglian type: in fact, there is no solution in which the small pearls are sold for an integral number of ducats and therefore also the large pearls.

§ 92. For the numbers 16, 48, 60, ... the systems are

[16, 48, 60]

$(13; \frac{1}{7} : 3; \frac{33}{7})$, $(12; \frac{1}{9} : 4; \frac{33}{9})$, $(11; \frac{1}{11} : 5; \frac{33}{11})$, ..., $(1; \frac{1}{31} : 15; \frac{33}{31})$;
$(11; \frac{2}{12} : 5; \frac{34}{12})$, $(10; \frac{2}{14} : 6; \frac{34}{14})$, $(9; \frac{2}{16} : 7; \frac{34}{16})$, ..., $(1; \frac{2}{32} : 15; \frac{34}{32})$;
$(9; \frac{3}{17} : 7; \frac{35}{17})$, $(8; \frac{3}{19} : 8; \frac{35}{19})$, $(7; \frac{3}{21} : 9; \frac{35}{21})$, ..., $(1; \frac{3}{33} : 15; \frac{35}{33})$;
$(7; \frac{4}{22} : 9; \frac{36}{22})$, $(6; \frac{4}{24} : 10; \frac{36}{24})$, $(5; \frac{4}{26} : 11; \frac{36}{26})$, ..., $(1; \frac{4}{34} : 15; \frac{36}{34})$;
$(5; \frac{5}{27} : 11; \frac{37}{27})$, $(4; \frac{5}{29} : 12; \frac{37}{29})$, $(3; \frac{5}{31} : 13; \frac{37}{31})$, ..., $(1; \frac{5}{35} : 15; \frac{37}{35})$;
$(3; \frac{6}{32} : 13; \frac{38}{32})$, $(2; \frac{6}{34} : 14; \frac{38}{34})$, $(1; \frac{6}{36} : 15; \frac{38}{36})$;
$(1; \frac{7}{37} : 15; \frac{39}{37})$.

Tartaglia's solution corresponds to the third system in the first line, viz. $(11; \frac{1}{11} : 5; 3)$; the other solutions in which the small pearls are sold for an integral number of ducats correspond to the third system in the third line and to the second system in the fourth line, viz. $(7; \frac{1}{7} : 9; 1\frac{2}{3})$ and $(6; \frac{1}{6} : 10; 1\frac{1}{2})$.

Partitionment into the form $\alpha + \beta$ such that $\lambda\alpha$ and $\mu\beta$ are integral, §§ 93–99.

§ 93. It is interesting to examine the conditions which are requisite in order that each portion of the fixed-sum may be integral, *i.e.* so that each part multiplied by its modulus may be integral.

The product of the first part and the first modulus is integral if the first part contains r as a factor and if the modulus is $\frac{1}{r}$, and also if the modulus in its lowest terms is $\frac{q}{r}$, where q is greater than unity and less than r; and the product of the second part and the second modulus is integral if the second modulus is an integer, or if the second part contains t

as a factor and the second modulus in its lowest terms is $\frac{s}{t}$.

The representations of a in the four cases are of the forms

$$\text{(i)}\ \left|kr;\ \frac{1}{r}:\ m;\ s\right|,\quad \text{(ii)}\ \left|kr;\ \frac{1}{r}:\ mt;\ \frac{s}{t}\right|,$$

$$\text{(iii)}\ \left|kr;\ \frac{q}{r}:\ m;\ s\right|,\quad \text{(iv)}\ \left|kr;\ \frac{q}{r}:\ mt;\ \frac{s}{t}\right|;$$

and the representations of $3a$, $5a$, ... are always of the same form as that of a; so that we may speak of a system or solution, as well as a representation, as being of one of these forms.

§ 94. Regarded as a question relating to the sale of pearls, the limitation is that an integral number of ducats is to be produced by the sale of the small pearls, and therefore also by the sale of the large pearls, and the four forms of the representation of a correspond to the cases in which the small pearls are sold at so many for a ducat, or at so many for q ducats, and the large pearls are each sold for an integral number of ducats, or each for an integral number of ducats and a fraction of a ducat.

Tartaglia's solutions are always of the form (i), but it is to be noticed that in the cases of $a=10$ and $a=11$ the question only admits of solutions of this form, viz. the representations of 10 and 11 are

$$10\mid 7;\ \tfrac{1}{7}:\ 3;\ 3\mid 10,\quad 11\mid 6;\ \tfrac{1}{6}:\ 5;\ 2\mid 11;$$

but for $a=16$, besides Tartaglia's solution of the form (i), viz. in which the representation of 16 is

$$16\mid 11;\ \tfrac{1}{11}:\ 5;\ 3\mid 16,$$

there are two in which it is of the form (ii), viz.

$$16\mid 6;\ \tfrac{1}{6}:\ 10;\ 1\tfrac{1}{2}\mid 16,\quad 16\mid 7;\ \tfrac{1}{7}:\ 9;\ 1\tfrac{2}{3}\mid 16.$$

§ 95. Taking α, β to be the two parts of the partition and λ, μ to be the moduli we see from the general formula (i) of § 83 that if $\lambda\alpha$ is to be integral, we must have, in the case of a, $\left(a-\frac{p-l}{2}\right)\frac{l}{p}$ integral, and in order that $\lambda\alpha$ may be integral also in the case of $3a$, $5a$, &c., $(2a+l)\frac{l}{p}$ must also be integral, $2a+l$ being the difference between consecutive numbers in the first column

of the system. It is sufficient to secure that $\lambda\alpha$ is integral, as then $\mu\beta$ is necessarily so too.

The conditions therefore are that $(2a+l-p)\,l$ is to be divisible by $2p$, and that $(2a+l)\,l$ is to be divisible by p: and it is to be remembered that l and p must be both even or both uneven.

If l is uneven the quotient of $(2a+l)\,l$ by p is uneven, $=g$ say, and therefore $(2a+l-p)\,l=(g-p)\,l$, which is necessarily divisible by 2, since q and p are uneven. In this case therefore the sole condition is that $(2a+l)\,l$ should be divisible by p.

If l is even, lp is certainly divisible by $2p$, and therefore the sole condition is that $(2a+l)\,l$ should be divisible by $2p$, that is, that $(2a+l)\,l_1$, where $l_1=\frac{1}{2}l$, should be divisible by p.

Thus both cases are included in the general statement that the two parts of the fixed-sum are integral for any value of p such that $(2a+l)\,l'$ is divisible by p, where l' denotes l when l is uneven and $\frac{1}{2}l$ when l is even.

§ 96. If then, for any value of a, we form for each possible value of l the value of $(2a+l)\,l'$, then all the admissible values of p which are divisors of $(2a+l)\,l'$ give systems in which each part of the fixed-sum is integral.

Taking for example $a=10$, if $l=1$, $(2a+l)\,l'=21\times 1$ and the admissible values of p are 7, 9, ..., 19, of which 7 alone divides 21×1, and therefore gives a system in which both parts of the fixed-sum are integral; if $l=2$, $(2a+l)\,l'=22\times 1$, the admissible values of p are 12, 14, ..., 20, none of which is a divisor of 22×1; if $l=3$, $(2a+l)\,l'=23\times 3$, the admissible values of p are 17, 19, 21, none of which divides 23×3; and if $l=4$, $(2a+l)\,l'=24\times 2$, and the only admissible value of p is 22. Thus the only representation of 10 for which the two parts are integral is $|7;\ \frac{1}{7}:\ 3;\ 3|$.

Taking as another example $a=16$, the procedure, more briefly expressed, is as follows:

$$
\begin{array}{lllll}
a=16, & l=1, & (2a+l)\,l'= & 33\times 1; & (\ 7,\ 31);\ p=11, \\
\ \ ,, & l=2, & \quad ,, & 34\times 1; & (12,\ 32), \\
\ \ ,, & l=3, & \quad ,, & 35\times 3; & (17,\ 33);\ p=21, \\
\ \ ,, & l=4, & \quad ,, & 36\times 2; & (22,\ 34);\ p=24, \\
\ \ ,, & l=5, & \quad ,, & 37\times 5; & (27,\ 35), \\
\ \ ,, & l=6, & \quad ,, & 38\times 3; & (32,\ 36), \\
\ \ ,, & l=7, & \quad ,, & 39\times 7; & (37),
\end{array}
$$

where the numbers in brackets are the smallest and largest admissible values of p (which are even or uneven according as l is even or uneven), and the values of p shown are those which are divisors of $(2a+l)\,l'$. Thus we obtain for 16 the following three representations in which the parts of the fixed-sum are integral

$$|11;\ \tfrac{1}{11}:\ 5;\ 3|,\quad |7;\ \tfrac{1}{7}:\ 9;\ 1\tfrac{2}{3}|,\quad |6;\ \tfrac{1}{6}:\ 10;\ 1\tfrac{1}{2}|.$$

The first of these representations is of the form (i) and the other two are of the form (ii), (§ 93).

§ 97. As an example in which all the forms (i), (ii), (iii), (iv) occur, I give the systems for $a=49$, in which both parts of the fixed-sum are integral. The values of l and $2a+l$ are given in square brackets and are written before the systems belonging to the value of l. As in §§ 90–92 only the first lines of the systems are given.

$$a=49.$$

$[\ 1,\ \ 99]$, $(45;\ \tfrac{1}{9}:\ 4;\ 11)$, $(44;\ \tfrac{1}{11}:\ 5;\ 9)$, $(33;\ \tfrac{1}{33}:\ 16;\ 3)$;

$[\ 2,\ 100]$, $(40;\ \tfrac{1}{10}:\ 9;\ 5)$, $(25;\ \tfrac{1}{25}:\ 24;\ 2)$;

$[\ 4,\ 102]$, $(34;\ \tfrac{2}{17}:\ 15;\ 3)$, $(17;\ \tfrac{1}{17}:\ 32;\ 1\tfrac{1}{2})$;

$[\ 6,\ 104]$, $(26;\ \tfrac{3}{26}:\ 23;\ 2)$, $(13;\ \tfrac{1}{13}:\ 36;\ 1\tfrac{1}{3})$;

$[\ 7,\ 105]$, $(28;\ \tfrac{1}{7}:\ 21;\ 2\tfrac{1}{7})$;

$[10,\ 108]$, $(27;\ \tfrac{5}{27}:\ 22;\ 2)$, $(24;\ \tfrac{1}{6}:\ 25;\ 1\tfrac{4}{5})$, $(9;\ \tfrac{1}{9}:\ 40;\ 1\tfrac{1}{5})$;

$[12,\ 110]$, $(22;\ \tfrac{2}{11}:\ 27;\ 1\tfrac{2}{3})$;

$[14,\ 112]$, $(\ 7;\ \tfrac{1}{7}:\ 42;\ 1\tfrac{1}{7})$.

Of these fifteen systems five are of the form (i), six of the form (ii), three of the form (iii), and one of the form (iv).

The systems can be completed at sight by successively adding the second number in square brackets (*i.e.* the value of $2a+l$) to the first number and subtracting the first number in square brackets (*i.e.* the value of l) from the third. Thus the systems derived from the first representation and that for $l=12$ are

49	45;	$\tfrac{1}{9}$:	4;	11	49,	49	22;	$\tfrac{2}{11}$:	27;	$1\tfrac{2}{3}$	49
147	144;	,,:	3;	,,	,,	147	132;	,,:	15;	,,	,,
245	243;	,,:	2;	,,	,,	245	242;	,,:	3;	,,	,, .

§ 98. The representations of a derived from $l=1$ and $l=2$ must always be of the form (i) *i.e.* $\left| kr;\ \frac{1}{r}:\ m;\ s \right|$; for, if $l=1$, the numerator of the first modulus must be unity, and if $l=2$, the numerator is 2, but the denominator p is also even, so that the factor 2 divides out and leaves unity as the numerator; and as the factor l' is in both cases unity, p has to be a divisor of $2a+l$, and therefore the second modulus is an integer.

A representation of the form (iv) is obtained when l is not a factor of p nor is p a factor of $2a+l$.

It is to be noted that the condition that p must be $>5l$ is due to the fact that the system is supposed to include three numbers at least, viz. a, $3a$, $5a$. If only two representations need to be included, viz. a, $3a$, the condition is that $p>3l$, and the values $3l+2$, $3l+4$, ..., $5l$ are then admissible for p.

§ 99. If we are given a representation of a in which the moduli are in their lowest terms, or at all events not given as $\frac{l}{p}$ and $\frac{2a+l}{p}$, and if it is required to complete the system, it is necessary to calculate l, from which of course $2a+l$ is at once derivable. To obtain l suppose that $\frac{l_0}{p_0}$ is the first modulus as given in the representation, and let h be the third number in the representation; then

$$\frac{l}{p}=\frac{l_0}{p_0},\quad \frac{p-l}{2}=h,$$

and therefore
$$l=\frac{2l_0h}{p_0-l_0}.$$

For example, in the case of the representation $\left|22;\ \frac{2}{11}:\ 27:\ 1\frac{2}{3}\right|$ of 49, we have $l_0=2$, $p_0=11$, $h=27$, whence $l=\frac{2\times 2\times 27}{9}=12$ and therefore $2a+l=110$.

General systems in which the two parts of the fixed-sum are integral, §§ 100–111.

§ 100. The most general representation of a for which the two parts of the fixed-sum are integral is

$$\left| kp;\ \frac{l}{p}:\ mt;\ \frac{s}{t} \right|.$$

Now, in general, if we put $2a+l=\frac{s}{t}p$ in (i) of § 83, the representation of a becomes

$$a \left| a-\frac{p-l}{2};\ \frac{l}{p}:\ \frac{p-l}{2};\ \frac{s}{t} \right| a,$$

and, since $a=\frac{sp-tl}{2t}$, this becomes

$$\frac{sp-tl}{2t} \left| \frac{(s-t)\,p}{2t};\ \frac{l}{p}:\ \frac{p-l}{2};\ \frac{s}{t} \right| \frac{sp-tl}{2t}.$$

If in this representation the two parts into which a is divided, and the two parts into which the fixed-sum is divided, are both integral, then this will be the case also in the representations of $3a$ and $5a$, for the two other lines of the system for a, $3a$, $5a$ are derived successively from the first line by adding $\frac{sp}{t}$, which is integral, in the first column and subtracting l in the third column. The condition that there should be representations of $3a$ and $5a$ (*i.e.* that the system should consist of at least three lines) is that $p>5l$.

§ 101. There is no loss of generality or alteration in the magnitude of the numbers represented if we replace $\frac{s}{t}$ by its equivalent fraction in its lowest terms $\frac{s_0}{t_0}$, where s_0 is prime to t_0. The representation of a then becomes

$$\frac{s_0p-t_0l}{2t_0} \left| \frac{(s_0-t_0)\,p}{2t_0};\ \frac{l}{p}:\ \frac{p-l}{2};\ \frac{s_0}{t_0} \right| \frac{s_0p-t_0l}{2t_0},$$

in which we know that $\frac{s_0p}{t_0}$ is integral, and we suppose that $p>5l$.

In order that the two parts into which a is partitioned may be integral we must have

$$\frac{(s_0-t_0)\,p}{2t_0} \text{ and } \frac{p-l}{2}$$

integral; and in order that the two parts of the fixed-sum may be integral we must have

$$\frac{(s_0-t_0)\,l}{2t_0} \text{ and } \frac{(p-l)\,s_0}{2t_0}$$

integral.

Taking the last of these four expressions, we must have

$$\frac{s_0p}{2t_0}-\frac{s_0l}{2t_0}=I,$$

where I is integral. Now, since $\frac{s_0 p}{t_0}$ is integral and s_0 is prime to t_0, it follows that p must be divisible by t_0, $=t_0 p_1$ say, and therefore

$$s_0 p_1 - \frac{s_0 l}{t_0} = 2I.$$

Since s_0 is prime to t_0, l must be divisible by t_0, $=t_0 l_1$ say, and the equation becomes $s_0 p_1 - s_0 l_1 = 2I$.

If s_0 is uneven, I must be divisible by s_0, $=I's_0$ say, and we have $p_1 - l_1 = 2I'$, so that in this case p_1 and l_1 must be both even or both uneven.

If s_0 is even, there is no restriction, and p_1 and l_1 may be either even or uneven. Replacing p and l by $s_0 p_1$ and $s_0 l_1$ the other expressions which have to be integral are

$$\frac{(s_0 - t_0)\, p_1}{2}, \quad \frac{(p_1 - l_1)\, t_0}{2}, \quad \frac{(s_0 - t_0)\, l_1}{2},$$

and it has been already shown that if s_0 is uneven, p_1 and l_1 must be both even or both uneven.

Taking the three cases of (i) s_0 uneven, t_0 uneven, (ii) s_0 uneven, t_0 even, (iii) s_0 even, t_0 uneven, these expressions show that

in case (i), p_1 and l_1 must be both even or both uneven;

in cases (ii) and (iii), p_1 and l_1 must be both even.

§ 102. It has thus been found that, if s_0 and t_0 are both uneven, p_1 and l_1 must be both even or both uneven, and that, if either s_0 or t_0 be even and the other uneven, then p_1 and l_1 must be both even: and conversely, if p_1 and l_1 are both uneven, s_0 and t_0 must be both uneven, and if p_1 and l_1 are both even, s_0 and t_0 may be both uneven, or one may be even and the other uneven. We thus obtain the following system in which p_1, l_1, s_0, t_0 are subject to these conditions and in which $p_1 > 5l_1$ and $s_0 > t_0$ and prime to t_0:

$\frac{s_0 p_1 - t_0 l_1}{2}$	$\frac{(s_0 - t_0)\, p_1}{2}$; $\frac{l_1}{p_1}$: $\frac{(p_1 - l_1)\, t_0}{2}$; $\frac{s_0}{t_0}$	$\frac{s_0 p_1 - t_0 l_1}{2}$	
3 („ „)	$\frac{(3s_0 - t_0)\, p_1}{2}$; „ : $\frac{(p_1 - 3l_1)\, t_0}{2}$; „	„	...(i).
5 („ „)	$\frac{(5s_0 - t_0)\, p_1}{2}$; „ : $\frac{(p - 5l_1)\, t_0}{2}$; „	„	

§ 103. For any given even values of l_1 and p_1 we can take any values of s_0 and t_0 (subject of course to the condition that s_0 is greater than t_0 and prime to it). If we put $l_1 = 2l'$ and $p_1 = 2p'$ we obtain the system

$$\begin{array}{l|l|l}
s_0 p' - t_0 l' & (s_0 - t_0)\, p' \;;\; \dfrac{l'}{p'} : \; (p' - \;l')\, t_0 \;;\; \dfrac{s_0}{t_0} & s_0 p_1 - t_0 l_1 \\
3(\;,,\quad ,,\;) & (3s_0 - t_0)\, p' \;;\; ,, : \; (p' - 3l')\, t_0 \;;\; ,, & \quad ,, \\
5(\;,,\quad ,,\;) & (5s_0 - t_0)\, p' \;;\; ,, : \; (p' - 5l')\, t_0 \;;\; ,, & \quad ,,
\end{array} \qquad \ldots \text{(ii)},$$

in which l' and p' may have any values (subject of course to l' being $< p'$) and s_0 and t_0 are as above.

From this general system (ii) we can obtain systems in which l' and p' are both uneven, but we also obtain other systems not included in (ii) by taking l_1 and p_1 in (i) to be uneven.

§ 104. If l' and p' have a common factor it appears as a factor in the numbers represented by the system, in the fixed-sum, and in the first and third columns of the table (and therefore in each of the parts of the number and fixed-sum). It can therefore be divided out from the numbers, the fixed-sum, and the first and third columns.

We may therefore take l' and p' to be prime to each other. The case in which l_1, p_1, s_0, t_0 are all uneven, and for which recourse is to be had to the system (i), may be included in the system (ii) if it is understood that when l', p', s_0, t_0 are all uneven the factor 2 which occurs in the numbers, fixed-sum, and in the first and third columns is to be divided out.

§ 105. I proceed now to deduce from (ii) the simplest of the systems in which the two parts of the fixed-sum are integral. The smallest admissible values of l', p' are 1, 6, and by taking $s_0 = 2$, $t_0 = 1$; $s_0 = 3$, $t_0 = 1$; $s_0 = 3$, $t_0 = 2$, ..., we obtain the following representations of the first numbers of systems. The number represented is as usual placed before the representation, and it is preceded by the values of s_0, t_0, which are placed in square brackets:

$[2, 1],\ 11 \,|\ 6 ; \tfrac{1}{6} : \ 5 ; 2 \,| ; [3, 1],\ 17 \,|\, 12 ; \tfrac{1}{6} : \ 5 ; 3 \,| ; [3, 2],\ 16 \,|\ 6 ; \tfrac{1}{6} : 10 ; \tfrac{3}{2}] ;$

$[4, 1],\ 23 \,|\, 18 ; \tfrac{1}{6} : \ 5 ; 4 \,| ; [4, 3],\ 21 \,|\ 6 ; \tfrac{1}{6} : 15 ; \tfrac{4}{3} \,| ; [5, 1],\ 29 \,|\, 24 ; \tfrac{1}{6} : \ 5 ; 5] ;$

$[5, 2],\ 28 \,|\, 18 ; \tfrac{1}{6} : 10 ; \tfrac{5}{2} \,| ; [5, 3],\ 27 \,|\, 12 ; \tfrac{1}{6} : 15 ; \tfrac{5}{3} \,| ; [5, 4],\ 26 \,|\ 6 ; \tfrac{1}{6} : 20 ; \tfrac{5}{4}] ;$

&c.

The fixed-sum is omitted, as it is always equal to the number represented. To complete any system we add $2s_0p'$ in the first column and subtract $2t_0l'$ in the third. The values of s_0, t_0 are given in the square brackets which precede the representation, and the values of l', p' are the numerator and denominator of the first modulus. Taking, for example, the third representation, $2s_0p' = 6 \times 6 = 36$ and $2t_0l' = 4 \times 1 = 4$ and the complete system is

$$\begin{array}{r|rlrl|l} 16 & 6; & \tfrac{1}{6}: & 10; & \tfrac{3}{2} & 16 \\ 48 & 42; & ,,: & 6; & ,, & ,, \\ 80 & 78; & ,,: & 2; & ,, & ,, \end{array}\ .$$

§ 106. The next smallest admissible values of l', p' are 1, 7, and, as in this case, both l' and p' are uneven, we note that when s_0 and t_0 are also both uneven, we are to divide out by 2, and the system, after division, is to be completed from its first line by adding s_0p' in the first column and subtracting t_0l' in the third. Thus for $s_0 = 3$, $t_0 = 1$, the formula (ii) gives $20 \mid 14;\ \tfrac{1}{7}:\ 6;\ 3 \mid 20$, which, dividing out by 2, becomes $10 \mid 7;\ \tfrac{1}{7}:\ 3;\ 3 \mid 10$, and the system is completed by adding $3 \times 7 = 21$ in the first column and subtracting $1 \times 1 = 1$ in the third.

The first lines of the systems for l', $p' = 1$, 7 corresponding to those for l', $p' = 1$, 6 in the preceding section are

$[2,1]$, $13 \mid 7;\tfrac{1}{7}:\ 6;2 \mid$; $[3,1]$, $10 \mid 7;\tfrac{1}{7}:\ 3;3 \mid$; $[3,2]$, $19 \mid 7;\tfrac{1}{7}:12;\tfrac{3}{2} \mid$;
$[4,1]$, $27 \mid 21;\tfrac{1}{7}:\ 6;4 \mid$; $[4,3]$, $25 \mid 7;\tfrac{1}{7}:18;\tfrac{4}{3} \mid$; $[5,1]$, $17 \mid 14;\tfrac{1}{7}:\ 3;5 \mid$;
$[5,2]$, $33 \mid 21;\tfrac{1}{7}:12;\tfrac{5}{2} \mid$; $[5,3]$, $16 \mid 7;\tfrac{1}{7}:\ 9;\tfrac{5}{3} \mid$; $[5,4]$, $31 \mid 7;\tfrac{1}{7}:24;\tfrac{5}{4} \mid$;

&c.

The representations for [3, 1], [5, 1], and [5, 3], as given by (ii), have been divided by 2, so that in these cases the systems are completed by adding and subtracting s_0p' and t_0l' instead of $2s_0p'$ and $2t_0l'$ as in the other representations.

§ 107. The smallest values of l', p', in which l' is even and p' uneven, are l', $p' = 2$, 11 and the corresponding representations are

$[2,1]$, $20 \mid 11;\tfrac{2}{11}:\ 9;2 \mid$; $[3,1]$, $31 \mid 22;\tfrac{2}{11}:\ 9;3 \mid$; $[3,2]$, $29 \mid 11;\tfrac{2}{11}:18;\tfrac{3}{2} \mid$;
$[4,1]$, $42 \mid 33;\tfrac{2}{11}:\ 9;4 \mid$; $[4,3]$, $38 \mid 11;\tfrac{2}{11}:27;\tfrac{4}{3} \mid$; $[5,1]$, $53 \mid 44;\tfrac{2}{11}:\ 9;5 \mid$;
$[5,2]$, $51 \mid 33;\tfrac{2}{11}:18;\tfrac{5}{2} \mid$; $[5,3]$, $49 \mid 22;\tfrac{2}{11}:27;\tfrac{5}{3} \mid$; $[5,4]$, $47 \mid 11;\tfrac{2}{11}:36;\tfrac{5}{4} \mid$;

&c.

It will be noticed that 49 is the lowest number in which, when represented in the form $\left| kp\,;\ \frac{l}{p} : mt\,;\ \frac{s}{t} \right|$, none of the quantities k, l, m, s, t are unity.

§ 108. The notation of the system (ii) in § 103 may be rendered more symmetrical if we denote the first modulus by $\frac{l_1}{p_1}$ and the second by $\frac{l_2}{p_2}$. The system then becomes

$$\begin{array}{r|llll|l} a & (\ l_2 - p_2)\,p_1; & \frac{l_1}{p_1}: & (p_1 - \ l_1)\,p_2; & \frac{l_2}{p_2} & a \\ 3a & (3l_2 - p_2)\,p_1; & ,,: & (p_1 - 3l_1)\,p_2; & ,, & ,, \\ 5a & (5l_2 - p_2)\,p_1; & ,,: & (p_1 - 5l_1)\,p_2; & ,, & ,, , \end{array}$$

where a is $l_2p_1 - l_1p_2$ and l_1, p_1, l_2, p_2 may have any values subject to the conditions that $p_1 > 5l_1$ and $l_2 > p_2$: but there is no loss of generality in supposing that l_1 is prime to p_1, and l_2 is prime to p_2. If l_1, p_1, l_2, p_2 are all uneven primes, the factor 2 may be divided out from a and from the first and third columns.

§ 109. Some particular cases may be noticed. If we take $p_2 = p_1$, and $l_2 = p_1 + l_1$, $l_2 = 2p_1 + l_1$, $l_2 = 3p_1 + l_1$, ..., we obtain the systems of which the first lines are

$$p^2 \left| \ lp\ ;\ \frac{l}{p}: \ (p-l)\,p;\ 1 + \frac{l}{p} \right| p^2,$$

$$2p^2 \left| \ (p+l)\,p;\ \frac{l}{p}: \ (p-l)\,p;\ 2 + \frac{l}{p} \right| 2p^2,$$

$$3p^2 \left| \ (2p+l)\,p;\ \frac{l}{p}: \ (p-l)\,p;\ 3 + \frac{l}{p} \right| 3p^2.$$

The first system may be completed by adding $2\,(p+l)\,p$ in the first column and subtracting $2lp$ in the third column; the second by adding $2\,(2p+l)\,p$ in the first column and subtracting $2lp$ in the third column, &c.

§ 110. Taking as an example l, $p = 1$, 6 and 1, 7 these representations give

$$36 \mid 6;\ \tfrac{1}{6}:\ 30;\ 1\tfrac{1}{6} \mid 36, \qquad 49 \mid 7;\ \tfrac{1}{7}:\ 42;\ 1\tfrac{1}{7} \mid 49,$$

$$72 \mid 42;\ \tfrac{1}{6}:\ 30;\ 2\tfrac{1}{6} \mid 72, \qquad 98 \mid 56;\ \tfrac{1}{7}:\ 42;\ 2\tfrac{1}{7} \mid 98,$$

$$108 \mid 78;\ \tfrac{1}{6}:\ 30;\ 3\tfrac{1}{6} \mid 108, \qquad 147 \mid 105;\ \tfrac{1}{7}:\ 42;\ 3\tfrac{1}{7} \mid 147,\ \&c.$$

In the representation for 98 the numerators and denominators of the two moduli are all uneven, viz. 1, 7, 15, 7, and therefore we may divide by 2, thus obtaining the representation $49 \mid 28;\ \frac{1}{7}:\ 21;\ 2\frac{1}{7} \mid 49$.

By taking $l, p=2, 11$ and $3, 16$ we have the representations

$121 \mid\ 22;\ \frac{2}{11}:\ 99;\ 1\frac{2}{11} \mid 121,\quad 256 \mid\ 48;\ \frac{3}{16}:\ 208;\ 1\frac{3}{16} \mid 256,$

$242 \mid 143;\ \frac{2}{11}:\ 99;\ 2\frac{2}{11} \mid 242,\quad 512 \mid 304;\ \frac{3}{16}:\ 208;\ 2\frac{3}{16} \mid 512$, &c.

§111. In these investigations (§§ 95–110) I have followed Tartaglia in supposing that the numbers partitioned are of the forms a, $3a$, $5a$, ..., but I have also worked out in some detail the more general case in which the numbers are a, $a+d$, $a+2d$, ..., and have considered special values of d besides $d=2a$. But these results I leave to a separate paper.

Tartaglia's question in which both small and large pearls are sold each for an integral number of ducats, §§ 112–114.

§ 112. Tartaglia also* gives a question in which there are nine sons, and they receive 10, 20, ..., 90 pearls and are each to bring back 100 ducats. The solution is that the small pearls are to be sold for 1 ducat each and the large pearls for 11 ducats each, the distribution of the pearls among the sons being as shown in the following system:

10	1;	1:	9;	11	100
20	12;	1:	8;	11	,,
30	23;	1:	7;	11	,,
...	...	...	...	...	...
90	89;	1:	1;	11	,, .

This is the only solution; but if it had been allowable that a son could have had only small pearls, a tenth son could have been provided with 100 pearls, the additional line of the system being

$100 \mid 100;\ \ 1:\ \ 0;\ \ 11 \mid 100,$

and the number of ducats brought back would have been the same as the number of pearls given to the tenth son.

* This is question 139 referred to in the note on p. 55.

§ 113. Tartaglia's question and solution may be generalised for a, $2a$, $3a$, ... eggs as shown in the following system:

a	k ;	$1:$	$a-k$;	$a+1$	$a(a-k)+a$
$2a$	$k+a+1$;	„ :	$a-k-1$;	„	„
$3a$	$k+2a+2$;	„ :	$a-k-2$;	„	„
...	....		....		...
$(a-k)a$	$(a-k)a-1$;	„ :	1 ;	„	„
$(a-k+1)a$	$(a-k+1)a$;	„ :	0 ;	„	„ ,

where k is any number less than a.

As particular cases, putting $a=10$, and $k=0, 1, 2$, we have

10	0 ; 1 : 10 ; 11	110,	10	1 ; 1 : 9 ; 11	100
20	11 ; „ : 9 ; „	„	20	12 ; „ : 8 ; „	„
30	22 ; „ : 8 ; „	„	30	23 ; „ : 7 ; „	„
...	....	...	...	....	...
100	99 ; „ : 1 ; „	„	90	89 ; „ : 1 ; „	„
110	110 ; „ : 0 ; „	„	100	100 ; „ : 0 ; „	„ ,

10	2 ; 1 : 8 ; 11	90
20	13 ; „ : 7 ; „	„
30	24 ; „ : 6 ; „	„
...	....	...
80	79 ; „ : 1 ; „	„
90	90 ; „ : 0 ; „	„ .

The first system containing 11 lines is the most symmetrical if the lines containing a zero are included: the second is Tartaglia's system, of 9 lines, if the last line is omitted: the succeeding systems each contain one line less; the system, for $k=8$, containing only two lines, if zeros are excluded.

§ 114. Tartaglia's question is interesting because each pearl is sold for an integral number of ducats, while in his other questions the small pearls were sold at so many to the ducat, and only the large ones for an integral number of ducats. In this he followed the egg question, in which at the

first sale the eggs were sold at so many for a penny, and only at the second sale was each egg sold for an integral number of pence.

It would seem to have been more natural that at each sale an egg should have been sold for a definite number of pence, *i.e.* that in the original question instead of following the system

$$\begin{array}{r|l|l} 10 & 1,\ 7:\ 3;\ 3 & 10, \\ 30 & 4,\ ,,:\ 2;\ ,, & ,, \\ 50 & 7,\ ,,:\ 1;\ ,, & ,, \end{array} \quad \text{that is} \quad \begin{array}{r|l|l} 10 & 7;\ \tfrac{1}{7}:\ 3;\ 3 & 10 \\ 30 & 28;\ ,,:\ 2;\ ,, & ,, \\ 50 & 49;\ ,,:\ 1;\ ,, & ,,\ , \end{array}$$

the solution should have been given by the system

$$\begin{array}{r|l|l} 10 & 7;\quad 1:\quad 3;\quad 21 & 70 \\ 30 & 28;\quad ,,:\quad 2;\quad ,, & ,, \\ 50 & 49;\quad ,,:\quad 1;\quad ,, & ,,\ , \end{array}$$

in which the eggs are sold at a penny and at 21 pence each. This really was the form of solution adopted in an even earlier form of the question as will now be seen.

PART IV.

LEONARDO PISANO'S RULES FOR PARTITIONING TWO NUMBERS SO THAT THE SUMS ARE THE SAME OR IN A GIVEN PROPORTION AND THE SOLUTIONS INTEGRAL. SYSTEMS IN WHICH THE SUMS ARE IN ARITHMETICAL PROGRESSION.

Treatment by Leonardo Pisano of a question in which there are two sales at two markets and the same money is brought back, §§ 115–137.

§ 115. In Part 7 of Chapter 12 of Leonardo Pisano's *Liber Abbaci* under the heading "De duobus hominibus, qui habuerunt poma"* the question proposed is: of two men, one had 10 apples, the other 30: and when they were both in one

* "Scritti di Leonardo Pisano" by Boncompagni, vol. i., *Liber Abbaci*, pp. 298–302 (Rome, 1857). The title-page of the *Liber Abbaci* is "Il liber abbaci di Leonardo Pisano pubblicato secondo la lezione del codice Magliabechiano C. I, 2616, *Badia Fiorentina*, n.° 73. da Baldassarre Boncompagni . . . ." The Liber begins "Incipit liber Abaci Compositus a leonardo filio Bonacij Pisano In Anno. M°cc°ij°" and occupies the whole of vol. i. (pp. 1–459). Cantor ("Geschichte der Mathematik", 2nd edition, vol. ii., p. 7) mentions other manuscripts of Leonardo's *Liber Abbaci* besides that printed by Boncompagni, which is the second or revised edition of his *Abbacus*: he shows that the date of this revised edition must lie between 1220 and 1250 and thinks there is no reason to doubt the accuracy of the words "compositus anno 1202 et correctus ab eodem anno 1228" which occurs on one of the manuscripts.

market, each sold of his apples I know not how many. But their price was the same, and when they came to another market they sold all they had left similarly at an equal price: and the first man had as much from his 10 apples as the second. What was the price in each market and how many apples did each sell in each market? The solution is as follows: Divide the number of apples of the first man, that is 10 into two parts, so that if the first part be subtracted from the number of apples of the other man, that is from 30, there may remain a number which can be integrally divided by the second part (qui integraliter diuidatur per secundam partem), and the quotient (quod ex dinisione peruenerit) will be the price of each apple sold in the second market; and because once the first part is subtracted from once 30, once a penny will be the price of each apple sold in the first market (et quia de semel 30 extrahitur semel prima pars, erit semel denarius 1 pretium uniuscuiusque pomi uenditi in primo foro). He then gives an example: let the two parts of 10 be 6 and 4; subtract 6 from 30 and 24 remains, which divided by the second part, that is 4, gives 6 for the price of an apple sold in the second market: so that, as the price of apples in the first market is 1 denarius, in the second market it is 6 denarii.

He then proceeds to find the number of apples sold in each market; and his rule is: Take any part you wish of the first part into which 10 was previously divided (6 in the case of the example) for the number of apples sold by the first man in the first market, then, if this part be subtracted from the previous first part (*i.e.* 6) the remainder is the number of apples sold by the second man in the second market; and the number of apples sold by the second man in the first market is obtained by subtracting this number from 30. For example: 6 being the first of the two parts into which 10 was divided, we may take for the number of apples sold by the first man in the first market any number less than 6, and if this number is subtracted from 6 the remainder is the number of apples sold by the second man in the second market.* Thus if the first man sells one apple in the first market, the second sells 5 in the second market, and if the first man sells 5 apples in the first market, the second sells 1 in the second market.

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* I have slightly varied Leonardo's description. His words are "Sed ut habeas poma utriusque fori, accipe ergo alibitum ex predictis 6 partem, qualem uis, pro pomis primi hominis uenditis in primo foro; et aliam partem, scilicet residuum, extrahe de 30; et quot remanebunt, erunt poma primi hominis uendita in primo foro". It is clear that in the last sentence 'primi hominis' should be 'secundi hominis'.

§ 116. Using symbols, Leonardo's procedure is to divide 10 into two parts, the first part being r, and the second $10-r$. He forms $30-r$ and finds a value of r such that $30-r$ is divisible by the second part, $10-r$: and this quotient gives the number of denarii paid for an apple in the second market if one denarius is the price paid in the first market. He takes for the number of apples sold by the first man in the first market any number α_1 less than r; and he asserts that $r-\alpha_1$ is the number of apples sold by the second man in the second market, so that, using the notation of the present paper, his system is

$$\left.\begin{array}{c|cccc|} 10 & \alpha_1; & 1: & \beta_1 & ; \dfrac{30-r}{10-r} \\ 30 & \alpha_2; & ,,: & r-\alpha_1 & ; \quad ,, \end{array}\right. ,$$

where r is any number less than 10, such that $\dfrac{30-r}{10-r}$ is integral, α_1 is any arbitrary number less than r, and the values of β_1 and α_2 are derivable from these data by subtraction since

$$\alpha_1+\beta_1=10, \quad \alpha_2+r-\alpha_1=30.$$

In Leonardo's examples, r is taken to be 6, and α_1 is 1 and 5, so that his systems are

$$\begin{array}{c|cccc|c} 10 & 1; & 1: & 9; & 6 & 55, \\ 30 & 25; & ,,: & 5; & ,, & ,, \end{array} \qquad \begin{array}{c|cccc|c} 10 & 5; & 1: & 5; & 6 & 35 \\ 30 & 29; & ,,: & 1; & ,, & ,,. \end{array}$$

§ 117. The only possible values of r are 5, 6, 8, 9 so that Leonardo's process gives the following solutions:

$$\begin{array}{c|cccc|l} 10 & \alpha_1 & ;\ 1: & 10-\alpha_1; & 5 & 50-4\alpha_1, \ (\alpha_1=1, 2, 3, 4), \\ 30 & 25+\alpha_1; & ,,: & 5-\alpha_1; & ,, & \quad ,, \\ 10 & \alpha_1 & ;\ 1: & 10-\alpha_1; & 6 & 60-5\alpha_1, \ (\alpha_1=1, 2, 3, 4, 5), \\ 30 & 24+\alpha_1; & ,,: & 6-\alpha_1; & ,, & \quad ,, \\ 10 & \alpha_1 & ;\ 1: & 10-\alpha_1; & 11 & 110-10\alpha_1, \ (\alpha_1=1, 2, 3, \ldots, 7), \\ 30 & 22+\alpha_1; & ,,: & 8-\alpha_1; & ,, & \quad ,, \\ 10 & \alpha_1 & ;\ 1: & 10-\alpha_1; & 21 & 210-20\alpha_1, \ (\alpha_1=1, 2, 3, \ldots, 8). \\ 30 & 21+\alpha_1; & ,,: & 9-\alpha_1; & ,, & \quad ,, \end{array}$$

§ 118. Leonardo divides 10, the smaller number of apples (*i.e.* those possessed by the first man), into two parts and

subtracts the first part from 30 and from 10, and if the former remainder is divisible by the latter, the partition of 10 into the two parts is admissible, and the quotient is the price in the second market. He then takes an arbitrary number, less than the first part into which 10 was divided, for the number of apples sold by the first man in the first market, and subtracts this number from the first part, the remainder being the number of apples sold by the second man in the second market.

Although he divides 10 into two parts, he does not make any direct use of the second part as such, though he speaks of the number $30-r$ as having to be divided by the second part, *i.e.* he does not point out that $30-r$ is equal to the sum of 20 and the second part, so that the division is always possible whenever the second part is a divisor of 20, the difference between the numbers of apples. As far as his procedure is concerned he might have taken for r any number less than 10, and, if this value was admissible, for α_1 any number less than r.*

§ 119. Expressed symbolically: if 10 is divided into two parts r and s, so that $r+s=10$: then if s is a divisor of 20, the price of an apple in the second market is $1+\frac{20}{s}$; and if α_1 is the number of apples sold by the first man in the first market, the number sold by the second man in the second market is $r-\alpha_1$. Thus the system is

$$\begin{array}{c|cccc|} 10 & \alpha_1 \quad ; & 1: & 10-\alpha_1; & 1+\frac{20}{s} \\ 30 & 30-r+\alpha_1; & ,,: & r-\alpha_1; & ,, \end{array},$$

where the sole conditions are that s is less than 10 and a factor of 20, and that α_1 is less than r.

* Later on in the manuscript (p. 300) Leonardo gives another example in which the numbers of apples possessed by the two men are 12 and 32. He proceeds "et uolo iterum, ut pretium primi fori sit denarius 1: diuides 12 in duas ut libet partes; et habeas primam pro summa pomorum, que primus uendidit in primo foro, et secundus in secundo: et abice eam de numero pomorum secundi hominis, scilicet de 32; residuum, quod remansit, diuide per secundam partem; et quod proueneit erit pretium unius pomi uenditi in secundo foro. Deinde accipe quotuis poma ex predicta prima parte, et habeas ea pro pomis uenditis a primo homine in primo foro; et que ex ipsa parte remanserint, habeas pro pomis uenditis a secundo in secundo foro". He then supposes 8 and 4 to be the first and second parts into which 12 is divided: he subtracts 8 from 32, leaving 24, which divided by 4 gives 6 as the price in the second market: he then divides 8 into two parts arbitrarily, as 5 and 3, taking 5 for the number of apples sold by the first man in the first market, so that 3 is the number sold by the second man in the second market. As another case, he divides 12 into the parts 7 and 5, and subtracting 7 from 32, the remainder is 25, which divided by 5 gives 5 as the price in the second market: and he divides 7 into two parts arbitrarily for the number sold by the first man in the first market, and by the second man in the second market.

§ 120. In general, if n_1 and n_2 are the numbers of apples which the two men have, and if r is any number less than n_1 such that $\frac{n_2-r}{n_1-r}$ is integral, then Leonardo's system is

$$\left.\begin{array}{l|lcll} n_1 & \alpha_1 & ; & 1: & n_1-\alpha_1\,; & \dfrac{n_2-r}{n_1-r} \\ n_2 & n_2-r+\alpha_1; & & ,,: & r-\alpha_1\,; & ,, \end{array}\right| ,$$

where α_1 is any number less than r.

If the system be expressed wholly in terms of s, where s is any divisor of d, the difference between n_2 and n_1, it takes the form

$$\left.\begin{array}{l|lcll} n_1 & \alpha_1 & ; & 1: & n_1-\alpha_1 \quad ; & 1+\dfrac{d}{s} \\ n_2 & \alpha_1+s+d\,; & & ,,: & n_1-\alpha_1-s\,; & ,, \end{array}\right| ,$$

where α_1 is any number less than n_1-s.

§ 121. The general solution of Leonardo's question may be expressed by

$$\left.\begin{array}{l|lcll} n_1 & \alpha_1 & ; & n_1-r: & n_1-\alpha_1\,; & n_2-r \\ n_2 & n_2-r+\alpha_1; & & ,,\quad: & r-\alpha_1\,; & ,, \end{array}\right| ,$$

as will be shown in § 144.

Leonardo seems to have desired that the first price should be unity, and therefore his second price was obtained by dividing n_2-r by n_1-r whenever it was so divisible. If the first price is to be 2, the second price is $\frac{2(n_2-r)}{n_1-r}$ if this number is integral. Similarly, if the first price is to be 3, the second price is the quotient $\frac{3(n_2-r)}{n_1-r}$ if this number is integral, and so on. Leonardo must almost certainly have had these solutions in his mind when he wrote (§ 115) "et quia de semel 30 extrahitur semel prima pars, erit semel denarius 1 pretium uniuscuiusque pomi uenditi in primo foro", for these words imply that if twice r were subtracted from twice n_2, then the price would be twice a denarius, &c.

§ 122. If the first price is λ, all the possible values of μ, the second price, are included in $\frac{\lambda(n_2-r)}{n_1-r}$, where r is any number such that this quotient is integral, the general system being

$$\left.\begin{array}{l|lcll} n_1 & \alpha_1 & ; & \lambda: & n_1-\alpha_1\,; & \dfrac{\lambda(n_2-r)}{n_1-r} \\ n_2 & n_2-\alpha_1+r\,; & & ,,: & r-\alpha_1\,; & ,, \end{array}\right| .$$

This system gives for μ not only all the values which can be derived from the case of $\lambda=1$ by multiplying both prices by λ, but also the values for which $\lambda(n_2-r)$ is divisible by n_1-r, although n_2-r is not so divisible.

Taking the values $n_1=10$, $n_2=30$, it has been shown (§ 117) that for $\lambda=1$ the values of μ, that is of the quotient of n_2-r divided by n_1-r, are 5, 6, 11, 21, and that these values of μ correspond to the values 5, 6, 8, 9 of r. For $\lambda=2$ we have for μ, besides the doubles of these numbers, the additional value $\mu=7$, since $2(30-2)$ is divisible by $10-2$, although $30-2$ is not so divisible; so that for $\lambda=2$ the values of μ are 7, 10, 12, 22, 42, which correspond to the values 2, 5, 6, 8, 9 of r. For $\lambda=3$ we have the additional value $\mu=13$, which is obtained as the quotient of $3(30-4)$ divided by $10-4$, and which therefore corresponds to $r=4$, and also the additional value $\mu=23$, which is obtained as the quotient of $3(30-7)$ divided by $10-7$, and which therefore corresponds to $r=7$. For $\lambda=4$ we have the additional value $\mu=14$, but this may be derived from the value $\mu=7$, which has been already obtained for $\lambda=2$; and so on.

§ 123. Proceeding in this manner, and including only the solutions in which λ and μ are prime to each other, we find the complete list of values of λ, μ, with the corresponding values of r, to be

λ,	μ	r	λ,	μ	r
1,	5	5	2,	7	2
1,	6	6	3,	13	4
1,	11	8	3,	23	7
1,	21	9	7,	27	3

§ 124. The system in § 122 can be more conveniently expressed by means of the letters s and d, where $s+r=n_1$ and $d=n_2-n_1$: it then becomes

$$\begin{array}{c|ccccc|}
n_1 & \alpha_1 & ; \ \lambda: & n_1-\alpha_1 & ; & \lambda+\dfrac{\lambda d}{s} \\
n_2 & \alpha_1+s+d\,; & ,, : & n_1-\alpha_1-s\,; & & ,,
\end{array},$$

in which the additional values of μ, *i.e.* those which are not

derivable from the case $\lambda=1$ by multiplication by λ, are given by the values of s which are divisors of λd but not of d.

§ 125. From the point of view of algebra, which expresses methods by formulæ, this latter form is the better; but Leonardo's form is equally good from the point of view of arithmetic, which expresses methods by rules. It takes as one of the data, and puts in direct evidence, the number r, which is the sum of the numbers of apples sold by the first man in the first market and by the second man in the second market, and which gives (by subtracting unity from it) the number of systems which correspond to the prices λ and $\frac{\lambda(n_2-r)}{n_1-r}$; for the smallest value of α_1 is 1 and the largest is $r-1$, so that the number of values of α_1 is $r-1$.*

For example, taking the table in § 123 for $r=5$, we have $\lambda, \mu=1, 5$, and there are four systems; for $r=2$ we have $\lambda, \mu=2, 7$, and there is but one system.

§ 126. Leonardo next proceeds (p. 298) to explain how solutions may be obtained when the amount of money received by each man for his sales (*i.e.* the fixed-sum) is given. As an example he supposes that the money required is 45. He has already obtained solutions in which the sums received are 55 and 35 (§ 116), viz.

$$\begin{array}{r|rrrr|l} 10 & 1; & 1: & 9; & 6 & 55, \\ 30 & 25; & ,,: & 5; & ,, & ,, \end{array} \qquad \begin{array}{r|rrrr|l} 10 & 5; & 1: & 5; & 6 & 35 \\ 30 & 29; & ,,: & 1; & ,, & ,,. \end{array}$$

Thus a difference of 20 in the sums received corresponds to a difference of 4 in the number of apples sold by the first man in the first market (*i.e.* in α_1): so that a change of 1 in α_1 corresponds to a change of 5 in the sum received. Now the sum to be received is to be 10 less than 55, and so the change required in α_1 is 2, and therefore the system is

$$\begin{array}{r|rrrr|l} 10 & 3; & 1: & 7; & 6 & 45 \\ 30 & 27; & ,,: & 3; & ,, & ,,. \end{array}$$

§ 127. This procedure supposes that the given amount of money is one of the fixed-sums which correspond to the given values of λ and μ. In the case of Leonardo's example, where $\lambda=1$ and $\mu=6$, the only possible values of the given amount

* The number s is the excess of the number of apples sold by the first man in the second market over the number sold by the second man in the second market.

are 40, 45, and 50, besides the limiting values 35 and 55, which he uses.*

§ 128. If the sum of money received is less than the number of apples possessed by the second man, the rule given is: multiply it by some number which will make it greater than the number of apples possessed by the second man and less than the greatest fixed-sum†: then adjust the sale to this fixed-sum (consolabis uenditionem in ipso numero) and divide the price by the number by which the sum of money received was multiplied. His example is: Suppose 20 is the given sum: double 20, making 40, which is 15 less than 55: therefore add 3 to α_1, giving the solution

$$\begin{array}{r|rrrr|l} 10 & 4\,; & 1: & 6\,; & 6 & 40 \\ 30 & 28\,; & ,,: & 2\,; & ,, & ,, \end{array},$$

whence, by halving the prices, we have

$$\begin{array}{r|rrrr|l} 10 & 4\,; & \tfrac{1}{2}: & 6\,; & 3 & 20 \\ 30 & 28\,; & ,,: & 2\,; & ,, & ,, \end{array}.$$

§ 129. Leonardo's procedure of multiplying the sum of money received by an integer so as to bring it within the range of the fixed-sums can be applied only to such sums as when multiplied by an integer become equal to one of the fixed-sums. Thus in his example of $\lambda = 1$, $\mu = 6$ the only values to which the procedure applies are the factors of the fixed-sums 35, 40, 45, 50, 55.

It seems clear that Leonardo intended the multiplier to be an integer, but if m be this multiplier the prices are $\dfrac{1}{m}$ and $\dfrac{6}{m}$, one of which must be fractional, and there seems to be but little gain in restricting m to be integral.

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* In a 2-line system in which the numbers are 10, 30 the only possible values of λ, μ are 1, 21; 1, 11; 3, 23; 1, 6; 1, 5; 3, 13; 7, 27; 2, 7; if zero values of α_1 and β_2 are inadmissible, and subject to this condition, the only possible values of the fixed-sum are 34, 35, 38, 40 (2), 42, 45, 46, 50 (3), 55, 60, 65, 70 (2), 80, 90 (2), 100 (2), 110 (3), 120, 130 (2), 150 (2), 170 (2), 190 (2), 210, 230, 250. The number in brackets after the fixed-sum indicates the number of systems in which it occurs, *e.g.* 40 is the fixed-sum for the two systems

$$\begin{array}{r|rrrr|l} 10 & 4\,; & 1: & 6\,; & 6 & 40, \\ 30 & 28\,; & ,,: & 2\,; & ,, & ,, \end{array} \qquad \begin{array}{r|rrrr|l} 10 & 7\,; & 1: & 3\,; & 11 & 40 \\ 30 & 29\,; & ,,: & 1\,; & ,, & ,, \end{array}.$$

These results follow from § 68.

† Leonardo calls the solution in which the fixed-sum is as large as possible the 'diffinitio maior' ('Nam diffinitionem maiorem dicimus, quando primus homo nendit unum tantum pomum in niliori foro', p. 299) and the 'diffinitio minor' (p. 300) is that in which the fixed-sum is least.

§ 130. Leonardo supposes that the sum of money received is less than the number of apples possessed by the second man, *i.e.* less than n_2. If he were confining himself to solutions in which λ and μ were given numbers, it would have seemed more natural to have taken the limit to be the smallest fixed-sum instead of n_2. The fixed-sums extend (when zero values of α_1 and β_2 are excluded) from $\lambda n_2 + \mu - \lambda$ to $\mu n_1 - (\mu - \lambda)$, both inclusive, the common difference being $\mu - \lambda$, and as Leonardo seems to confine himself to solutions in which $\lambda = 1$, they extend from $n_2 + \mu - 1$ to $\mu n_1 - (\mu - 1)$, the common difference being $\mu - 1$. Thus for all sets of solutions in which $\lambda = 1$ the number of apples possessed by the second man, viz. n_2, is the number next below the least fixed-sum in the arithmetical progression to which the fixed-sums belong (*i.e.* in which the common difference is $\mu - 1$): being in fact the fixed-sum corresponding to $\beta_2 = 0$ in all the sets of systems, if zero values be admitted.

It might be thought that Leonardo selected n_2 as his limit because it was the same for all prices (if $\lambda = 1$), and it may have been so, but this is not an adequate reason for excluding n_2. It is true that in Leonardo's example of $\lambda = 1$, $\mu = 6$, 30 cannot be multiplied by an integral factor which will bring it within the range of the fixed-sums, but it can be brought within the range of the systems for which $\lambda = 1$, $\mu = 11$ and $\lambda = 1$, $\mu = 21$ by multiplication by 2 and 3, the final results being

10	5 ;	$\frac{1}{2}$:	5 ;	$\frac{11}{2}$	30,		10	6 ;	$\frac{1}{3}$:	4 :	7	30
30	27 ;	,, :	3 ;	,,	,,		30	27 ;	,, :	3 ;	,,	,, .

§ 131. If the amount of money received exceeds the largest fixed-sum, he multiplies the latter by a factor which will make it larger than the former: he then proceeds as before, subtracting the given fixed-sum from the new fixed-sum and dividing the remainder by the difference between the prices, and also by the factor which has been used, and applies the number so obtained to the α_1 of the system with the largest fixed-sum: he then multiplies the prices by the same factor. For example, if the required fixed-sum is to be 100, he doubles the largest fixed-sum 55, obtaining 110: the difference is 10, which divided by 5 (the difference of the prices) gives 2: this divided by the factor 2 (by which 55 was multiplied) gives 1: which, added to α_1 in the system in which the fixed-sum was 55, gives 2; whence, doubling the prices, we have the system

10	2 ;	2 :	8 ;	12	100
30	26 ;	,, :	4 ;	,,	,, .

§ 132. Leonardo also points out that we may proceed in another manner so as to obtain any given number as fixed-sum; for in any solution the fixed-sum can be altered in any proportion by altering the prices in this proportion.* His example is that if the fixed-sum is to be 70, by taking the solution in which the fixed-sum is 35 (§ 116 or 126) and multiplying the prices by 2 we obtain the solution

$$\begin{array}{c|ccccccc|c} 10 & 5\,; & & 2: & & 5\,; & & 12 & 70 \\ 30 & 29\,; & & ,,: & & 1\,; & & ,, & ,, \end{array}\;.$$

§ 133. This procedure is quite general and is applicable in all cases, whatever the amount of money received may be. If σ' denote the amount of money received it is only necessary to take any system

$$\begin{array}{c|ccccccc|c} n_1 & \alpha_1\,; & & \lambda: & & \beta_1\,; & & \mu & \sigma \\ n_2 & \alpha_2\,; & & ,,: & & ,,\,; & & ,, & ,, \end{array}$$

in which the two sums are the same, and replace λ and μ by λ' and μ', where λ' and μ' bear the same ratio to λ and μ that σ' bears to σ.

The solutions described in §§ 126 and 131 are of only partial application, and those in § 128 involve fractional prices and so present no advantage over those given by this general procedure, which applies to all the cases treated of in §§ 126, 128, and 131.

§ 134. Leonardo next considers cases in which the sum received by one man is a multiple of that received by the other, after which he appends a supplementary section relating to the case of equal sums, which it is convenient to describe here, in immediate connection with his previous treatment of the same question. An account of his procedure when one sum is to be a multiple of the other is given in §§138–141.

§ 135. In this additional section which has a special heading 'Modus alius in questione pomorum', Leonardo considers the possibility and mode of solving the question when the prices are given or their ratio. His rule is: Multiply the smaller price by the greater number of apples, and the greater price by the smaller number of apples: and if the latter

* "Potes etiam alio modo ad habendam quamlibet summam denariorum procedere, cum solidata erunt poma utriusque in aliquo denariorum numero; quia erit sicut numerus ille ad summam quesitam, ita pretium inuentum uniuscuiusque fori ad pretium quesitum eiusdem" (p. 299).

product is the greater the question can be solved. In this case, subtract the smaller product from the greater, and from the remainder subtract the difference between the prices, once, twice, or as often as you wish, so long as something remains: and this remainder, divided by the differences between the prices, may be taken for the number of apples sold by the second man in the second market, and as often as the difference of prices has been subtracted, so many apples sold the first man in the first market. As an example, he supposes the ratio of the prices to be as 1 to 4, and that the first man has 12 apples and the second 33. Since 4 times $12 = 48$ is greater than 1 times $33 = 33$, the problem admits of solution. Subtract 33 from 48, and from the remainder 15 subtract 3, the difference of the prices twice, leaving 9, which divided by the difference of the prices gives 3 as quotient: this is the number of apples sold by the second man in the second market,* and the number sold by the first man in the first market is 2, because the difference of the prices was subtracted twice. The system thus is

$$\begin{array}{c|cccc|c} 12 & 2\ ; & 1: & 10\ ; & 4 & 42 \\ 33 & 30\ ; & ,,: & 3\ ; & ,, & ,, \end{array} .$$

§ 136. He also supposes that the number of apples sold by the first man in the first market is given, viz. 3: he therefore sells 9 in the second market and receives altogether 39 denarii: he subtracts once 33 from 39, and divides the remainder by the difference of the prices, 3, giving 2, which is the number of apples sold by the second man in the second market, the system being

$$\begin{array}{c|cccc|c} 12 & 3\ ; & 1: & 9\ ; & 4 & 39 \\ 33 & 31\ ; & ,,: & 2\ ; & ,, & ,, \end{array} .$$

§ 137. Expressed symbolically, Leonardo's question is: if the numbers of apples n_1, n_2 are given and also the prices λ, μ, does a solution exist? and the answer is that there is a solution if $\mu n_1 > \lambda n_2$. To obtain the solution when this condition is satisfied, he forms the expression $\mu n_1 - \lambda n_2 - m(\mu - \lambda)$, where m is arbitrary subject only to the condition that the expression is positive, and then the expression divided by $\mu - \lambda$ gives β_2, the value of α_1 being m, *i.e.*

$$\alpha_1 = m, \quad \beta_2 = \frac{\mu n_1 - \lambda n_2}{\mu - \lambda} - m.$$

* There is some omission in the text, but the meaning is clearly as I have given it.

If zero values are excluded, as Leonardo assumes, m must be at least equal to unity, and therefore $\mu n_1 - \lambda n_2$ must be greater than $\mu - \lambda$.

He next supposes (§ 136) that α_1 is given, and from it he forms the fixed-sum $\lambda\alpha_1 + \mu(n - \alpha_1)$ from which he subtracts λn_2, leaving $\mu n_1 - \lambda n_2 - (\mu - \lambda)\alpha_1$, which divided by $\mu - \lambda$ gives β_2 as before.

It will be noted that $\dfrac{\mu n_1 - \lambda n_2}{\mu - \lambda}$ is the r of §§ 116–125, and that Leonardo assumes that it is integral, which of course is not necessarily the case, and indeed it is only so when $\mu - \lambda$ is a divisor $n_2 - n_1$. He might have calculated its value and then, as before, have taken α_1 to be any number less than it.

Leonardo's treatment of a similar question when one of the sums brought back is a multiple of the other, §§ 138–143.

§ 138. Leonardo passes directly from the general procedure described in § 132 to questions in which the sum received by the first man is to be, not the same as, but a multiple of, that received by the second man. Using as before α_1, β_1 to denote the numbers of apples sold by the first man in the two markets, and α_2, β_2 to denote the numbers sold by the second man in the two markets, the rule, if the first sum is to be double of the second, is to start with any system in which the sums received are the same, and in which β_1 is greater than the double of β_2: then to subtract $2\beta_2$ from β_1 and divide the original fixed-sum* by the remainder. The quotient so obtained is to be added to the second price, and in the system so formed the first man will receive twice as much as the second man.

As an example, he starts with the system

12	6 ;	1 :	6 ;	5	36
32	31 ;	,, :	1 ;	,,	,, ;

he subtracts 2 from 6, leaving 4, and divides 36 by 4, giving 9 ; which added to the second price 5 makes 14, the required system being

12	6 ;	1 :	6 ;	14	90
32	31 ;	,, :	1 ;	,,	45.

* The words in the text (p. 300) are 'duplica predictam summam : de qua duplicatione abice ipsam summam, hoc est multiplica ipsam summam per 1, scilicet per unum, minus de 2'. Thus he forms his dividend as $2\sigma - \sigma = \sigma$, indicating that in general, if the first sum is to be k times the second, the dividend is $k\sigma - \sigma, = (k - 1)\sigma$.

§ 139. Similarly if the first sum is to be three times the second, $3\beta_2$ is to be subtracted from β_1 and twice the fixed-sum is to be divided by the remainder: and the quotient is to be added to the second price. For example, starting with the same system as before, in which both the sums are 36, the remainder obtained by subtracting 3 from 6 is 3, which divided into twice 36, that is into 72, gives the quotient 24, which added to 5 is 29, so that the system is

$$\begin{array}{c|cccc|c} 12 & 6\ ; & 1: & 6\ ; & 29 & 180 \\ 32 & 31\ ; & ,,: & 1\ ; & ,, & 60, \end{array}$$

and a similar procedure is to be followed if the first sum is to be four times the second; and so on.

§ 140. The general process may be expressed as follows. In the system

$$\begin{array}{c|cccc|c} n_1 & \alpha_1\ ; & \lambda: & \beta_1\ ; & \mu & \sigma \\ n_2 & \alpha_2\ ; & ,,: & \beta_2\ ; & ,, & ,, \end{array}$$

suppose that $\beta_1 > k\beta_2$ and add to μ the quantity

$$\frac{(k-1)\,\sigma}{\beta_1 - k\beta_2}.$$

The first sum then becomes

$$\sigma + \frac{(k-1)\,\beta_1\sigma}{\beta_1 - k\beta_2}, \quad = \frac{k\,(\beta_1 - \beta_2)\,\sigma}{\beta_1 - k\beta_2};$$

and the second sum becomes

$$\sigma + \frac{(k-1)\,\beta_2\sigma}{\beta_1 - k\beta_2}, \quad = \frac{(\beta_1 - \beta_2)\,\sigma}{\beta_1 - k\beta_2},$$

the first sum being therefore k times the second.

§ 141. Leonardo then considers the case in which the money received by the second man is to be a multiple of that received by the first. As an example, he supposes that the second man is to receive four times as much as the first: he starts with a system in which both sums are the same, and subtracts $\frac{1}{4}\beta_2$ from β_1: he divides $\frac{3}{4}$ of the fixed-sum by the remainder, and subtracts the quotient from the second price. In his numerical example he starts with the system

$$\begin{array}{c|cccc|c} 12 & 6\ ; & 1: & 6\ ; & 11 & 72 \\ 32 & 28\ ; & ,,: & 4\ ; & ,, & ,, \end{array},$$

subtracts $\frac{1}{4}$ of 4, that is 1, from 6, leaving 5; he divides $\frac{3}{4}$ of 72, that is 54, by 5, giving $10\frac{4}{5}$, which he subtracts from 11, the remainder being $\frac{1}{5}$; and the system is

$$\begin{array}{c|ccccccc|c} 12 & 6\,; & 1 & : & 6\,; & \frac{1}{5} & & \frac{36}{5} \\[1ex] 32 & 28\,; & ,, & : & 4\,; & ,, & & \frac{144}{5} \end{array}\;,$$

and in order that the solution may be expressed by integers he multiplies each price by 5, and obtains the solution*

$$\begin{array}{c|ccccccc|c} 12 & 6\,; & 5 & : & 6\,; & 1 & & 36 \\[1ex] 32 & 28\,; & ,, & : & 4\,; & ,, & & 144. \end{array}$$

§ 142. Expressed as in § 140, the process is to subtract

$$\frac{(k-1)\sigma}{k\beta_1-\beta_2}$$

from the second price, so that the first sum becomes

$$\sigma-\frac{(k-1)\beta_1\sigma}{k\beta_1-\beta_2}, \quad =\frac{(\beta_1-\beta_2)\sigma}{k\beta_1-\beta_2},$$

and the second becomes

$$\sigma-\frac{(k-1)\beta_2\sigma}{k\beta_1-\beta_2}=\frac{k(\beta_1-\beta_2)\sigma}{k\beta_1-\beta_2}.$$

§ 143. In the questions described in §§ 138, 139, and 141 (where the sum received by one man is a multiple of that received by the other), Leonardo seems merely to have desired to form a single system in which n_1 and n_2, the numbers of apples possessed by the two men, were given, and the sums received were to be in a given ratio—*i.e.* he does not seem to have desired to show how all such solutions could be obtained, or to have subjected them to any other condition. He always seems to have desired that one of the prices, λ or μ, should be unity, the other being an integer. Thus in §§ 138 and 139 he selected, as an equal-sum system to start with, one in which $\beta_1-k\beta_2$ was a divisor of σ the fixed-sum. In § 141 the adjustment in order that μ should be unity was much more difficult (see § 167). There does not seem to be any reason why he should have formed $\beta_1-\frac{1}{4}\beta_2$ and divided $\frac{3}{4}\sigma$ by it rather than have formed $4\beta_1-\beta_2$ and divided 3σ by it, except that in his example β_2 and σ were divisible by 4; but this was not an essential condition, for starting with the system

$$\begin{array}{c|ccccccc|c} 12 & 7\,; & 1 & : & 5\,; & 11 & & 62 \\[1ex] 32 & 29\,; & ,, & : & 3\,; & ,, & & ,, \end{array}\;,$$

* Leonardo gives the sums as 3 soldi and 12 soldi, so that a soldus is 12 denarii.

and following Leonardo's procedure, we obtain the system

$$\begin{array}{c|cccc|c} 12 & 7\,; & 17: & 5\,; & 1 & 124 \\ 32 & 29\,; & ,,: & 3\,; & ,, & 496 \end{array} \quad .$$

Mathematical treatment of the question when the sums are equal, §§ 144–154.

§ 144. In the general mathematical question in which n_1, n_2 are the given numbers and λ, μ the moduli, and the sums are the same, the system is

$$\begin{array}{c|cccc|c} n_1 & \alpha_1\,; & \lambda: & \beta_1\,; & \mu & \sigma \\ n_2 & \alpha_2\,; & ,,: & \beta_2\,; & ,, & ,, \end{array},$$

where $$n_1 = \alpha_1 + \beta_1, \quad n_2 = \alpha_1 + \beta_2,$$

and $$\lambda\alpha_1 + \mu\beta_1 = \lambda\alpha_2 + \mu\beta_2 = \sigma.$$

Substituting for α_2 and β_1 in the last equation, we find

$$\lambda\alpha_1 + \mu(n_1 - \alpha_1) = \lambda(n_2 - \beta_2) + \mu\beta_2,$$

whence it follows that

$$\mu(n_1 - \alpha_1 - \beta_2) = \lambda(n_2 - \alpha_1 - \beta_2),$$

and therefore $\lambda : \mu :: n_1 - \alpha_1 - \beta_2 : n_2 - \alpha_1 - \beta_2$,

so that the system is

$$\begin{array}{c|cccc|c} n_1 & \alpha_1\,; & n_1 - \alpha_1 - \beta_2: & n_1 - \alpha_1\,; & n_2 - \alpha_1 - \beta_2 & n_1\alpha_2 - n_2\alpha_1 \\ n_2 & n_2 - \beta_2\,; & ,, : & \beta_2\,; & ,, & ,, \quad ,, \end{array} .$$

Let $\alpha_1 + \beta_2 = r$, then this system becomes

$$\begin{array}{c|cccc|} n_1 & \alpha_1\,; & n_1 - r: & n_1 - \alpha_1\,; & n_2 - r \\ n_2 & n_2 - r + \alpha_1\,; & ,, : & r - \alpha_1\,; & ,, \end{array} \ldots\ldots\text{(i)},$$

where r may have the values $0, 1, 2, \ldots, n_1 - 1$, and for any value r' of r, the admissible values of α_1 are $0, 1, 2, \ldots, r'$. If zero values of α_1 and β_2 are excluded, r has the values $2, \ldots, n_1 - 1$, and corresponding to any value r' of r, the values of α_1 are $1, 2, \ldots, r' - 1$.

§ 145. By multiplying the moduli in (i) by λ we obtain the system

$$\begin{array}{c|cccc|} n_1 & \alpha_1\,; & \lambda(n_1 - r): & n_1 - \alpha_1\,; & \lambda(n_2 - r) \\ n_2 & n_2 - r + \alpha_1\,; & ,, : & r - \alpha_1\,; & ,, \end{array},$$

and if r is such that n_1-r is a divisor of $\lambda(n_2-r)$, by dividing the moduli by n_1-r we obtain the system

$$\begin{array}{c|lcl|l} n_1 & \alpha_1 \;; & \lambda: & n_1-\alpha_1\;; & \dfrac{\lambda(n_2-r)}{n_1-r} \\ n_2 & n_2-r+\alpha_1\;; & ,,: & r-\alpha_1\;; & ,, \end{array} \;\ldots\ldots\text{(ii)},$$

in which λ is any given integer and the second modulus is also an integer.

It is clear that this general solution was known to Leonardo (§ 121), although in all his examples λ is taken to be 1.

§ 146. If we denote the second modulus by μ, then

$$\mu=\frac{n_2-r}{n_1-r}\lambda,$$

and therefore $$r=\frac{\mu n_1-\lambda n_2}{\mu-\lambda}.$$

The last system thus becomes

$$\begin{array}{c|lcl|l} n_1 & \alpha_1 \;; & \lambda: & n_1-\alpha_1 \;, & \mu \\ n_2 & \alpha_1+\dfrac{\mu(n_2-n_1)}{\mu-\lambda}\;; & ,,: & \dfrac{\mu n_1-\lambda n_2}{\mu-\lambda}-\alpha_1\;; & ,, \end{array}\;,$$

which may be expressed more symmetrically in the form

$$\begin{array}{c|lcl|l} n_1 & \alpha_1 \;; & \lambda: & n_1-\alpha_1 \;; & \mu \\ n_2 & \alpha_1+\dfrac{\mu(n_2-n_1)}{\mu-\lambda}\;; & ,,: & n_1-\alpha_1-\dfrac{\lambda(n_2-n_1)}{\mu-\lambda}\;; & ,, \end{array}\;\ldots\text{(iii)},$$

where λ and μ are any numbers such that $\lambda(n_2-n_1)$ is divisible by $\mu-\lambda$, and that the quotient does not exceed n_1.

§ 147. If in § 144 instead of substituting for α_2 and β_1 we had substituted for α_1 and α_2, we should have obtained the equation

$$\mu(\beta_1-\beta_2)=\lambda(n_2-n_1+\beta_1-\beta_2),$$

so that putting $\beta_1-\beta_2=s$, and $n_2-n_1=d$, we have

$$\lambda:\mu::s:s+d,$$

and the system becomes

$$\begin{array}{c|lclc|c} n_1 & n_1-\beta_1\;; & s: & \beta_1\;; & s+d & n_2\beta_1-n_1\beta_2 \\ n_2 & n_2-\beta_2\;; & ,,: & \beta_2\;; & ,, & ,, \end{array}\;\ldots\text{(iv)},$$

where β_1 and β_2 are subject only to the conditions that $\beta_1 \leq n_1$, $\beta_2 < \beta_1$; and $s = \beta_1 - \beta_2$, $d = n_2 - n_1$.

§ 148. This result may be expressed in various forms: for example, if expressed in terms of α_1, s, and d, it becomes

$$\left.\begin{array}{l|lcl}n_1 & \alpha_1 \quad ; & s: & n_1-\alpha_1 \quad ; \quad s+d \\ n_1+d & \alpha_1+s+d; & ,,: & n_1-\alpha_1-s; \quad ,, \end{array}\right| \ldots(\text{v}).$$

In this system s may have the values 1, 2, ..., n_1, and for any value s' of s the admissible values of α_1 are 0, 1, 2, ..., $n_1 - s'$. If zero values are excluded the admissible values of s are 1, 2, ..., $n_1 - 2$, and of α_1 are 1, 2, ..., $n_1 - s' - 1$.

§ 149. The system can also be expressed in the symmetrical form

$$\begin{array}{l|lcll|l}n_1 & \alpha_1; & \beta_1-\beta_2: & \beta_1; & \alpha_2-\alpha_1 & \alpha_2\beta_1-\alpha_1\beta_2 \\ n_2 & \alpha_2; & ,, \quad : & \beta_2; & ,, & ,, \end{array},$$

where the conditions are the same as in § 147.

§ 150. From § 148 we deduce that

$$\left.\begin{array}{l|lcll}n_1 & \alpha_1 \quad ; & \lambda: & n_1-\alpha_1 \quad ; & \lambda+\dfrac{\lambda d}{s} \\ n_2 & \alpha_1+s+d; & ,,: & n_1-\alpha_1-s; & ,, \end{array}\right| \ldots(\text{vi}),$$

which is the s-system which was derived in § 124 from the system in § 122.

If we denote the second modulus by μ, then

$$s = \frac{\lambda d}{\mu - \lambda},$$

and the system becomes

$$\left.\begin{array}{l|lcll}n_1 & \alpha_1 \quad ; & \lambda: & n_1-\alpha_1 \quad ; & \mu \\ n_2 & \alpha_1+\dfrac{\mu d}{\mu-\lambda}; & ,,: & n_1-\alpha_1-\dfrac{\lambda d}{\mu-\lambda}; & ,, \end{array}\right|,$$

which is the same as (iii) of § 146.

§ 151. If it is required to find all the systems corresponding to any given values of n_1 and n_2, the solution is conveniently supplied by (i) of § 144 or by (iv) of § 147. Taking the latter form, we give to s successively the values 1, 2, ..., n, and for each value of s' of s we give α_1 the values of 0, 1, 2, ..., $n_1 - s'$.

If systems containing zero values are excluded, the values n and $n-1$ of s are to be excluded, and also the values 0 and $n_1 - s'$ of α_1.

§ 152. But if the question is to find all the systems which correspond to a given value of λ, when n_1 and n_2 are given, then we have recourse to (ii) of § 145 or (v) of § 148. Taking the latter form, the values of s are all the numbers up to n inclusive which are divisors of λd; and for any such divisor s' the values of α_1 are $0, 1, 2, \ldots, n_1 - s'$: but if the systems with zero values are excluded, s must not have the values n or $n-1$, nor α_1 the values 0 and $n-s'$.

Among the solutions so obtained those that are derived from values of s which are divisors of d are merely multiples of those for which the first modulus is 1, those derived from values of s which are divisors of $\lambda_1 d$ (where λ_1 is a divisor of λ) are multiples of those for which the first modulus is λ_1. Thus if s is a divisor of λd, and not of $\lambda' d$, where λ' is any divisor of λ, then λ is prime to μ.

For example, taking the case of $n_1 = 10$, $n_2 = 30$, the values of μ for $\lambda = 1$ are given by $1 + \dfrac{20}{s}$, where s is any divisor of 20 which does not exceed 10, and therefore the values of s are 1, 2, 4, 5, 10, which give 21, 11, 6, 5, 3 as the values of μ. For $\lambda = 2$ the only value of s which is a divisor of 40 and not of 20 is 8, and $2 + \frac{40}{8}$, $= 7$, is the corresponding value of μ. Thus for $\lambda = 2$ the values of μ are 42, 22, 12, 10, 6, which are the doubles of the values of μ for $\lambda = 1$, and the new value 7. For $\lambda = 3$ the values of μ are the triples of the values of μ for $\lambda = 1$, viz. 63, 33, 18, 15, 9, which correspond to the divisors of 20; and corresponding to 3 and 6 which are divisors of 60 but not of 20 we have $3 + \frac{60}{3}$ and $3 + \frac{60}{6}$, that is 23 and 13 as values of μ. The values of μ corresponding to new divisors are always prime to λ.

If we take $k = 6$, the values of s for which μ, $= 6 + \dfrac{120}{s}$, is integral are 1, 2, 3, 4, 5, 6, 8, 10, giving $\mu = 126$, 66, 46, 36, 30, 26, 21, 18, of which 126, 66, 36, 30, 18 are values of μ for $\lambda = 1$ multiplied by 6; 21 is the value for $\lambda = 2$ multiplied by 3; and 46, 26 are the values for $\lambda = 3$ multiplied by 2.

If we take $\lambda = 7$, $\mu = 7 + \dfrac{140}{s}$, and the values of s are the divisors of 20 and the new divisor 7 which gives $\mu = 27$. Thus the values of μ are 147, 77, 42, 35, 21, and 27.

If zero values are excluded the value 10 for s is not admissible.

§ 153. The system (v) of § 148 can obviously be extended to higher numbers n_3, n_4, ..., if $n_3 - n_2 = d$, $n_4 - n_3 = d$, ..., that is, if the numbers n_1, n_2, n_3, n_4, ... are in arithmetical progression, with d as the common difference, unless s is too large, viz. dropping the suffix 1 from n_1 and α_1, we have from (v)

$$\left.\begin{array}{l|lllll} n & \alpha \quad ; & s: & n-\alpha & ; & s+d \\ n+d & \alpha+\ s+d\ ; & ,, : & n-\alpha-s & ; & ,, \\ n+2d & \alpha+2s+2d; & ,, : & n-\alpha-2s; & & ,, \\ \ldots\ldots & \ldots\ldots\ldots\ldots & & & & \end{array}\right| ,$$

the fixed sum being $ns + (n-\alpha)d$. The system can be continued so long as the numbers in the third column remain positive, *i.e.* there are in all $1 + I\left(\frac{n-\alpha}{s}\right)$ lines in the system.

This extension also applies to (vi) of § 150, the number of lines being the same.

Thus, in order that, for any given value of s, a system containing three or more lines can be derived from (v) or (vi), s must be $<$ or $= \frac{1}{2}n$.

The complete series of 3-line and 2-line systems for $n=10$, $d=20$ was given in § 68 (p. 49).

§ 154. Leonardo does not mention the extension to more numbers than two. It would not suggest itself, however, unless the quantities which involve n_1, n_2, λ, μ were expressed in such a form as in (iii) of § 146, where n_2 occurs in the second line only in the form $n_2 - n_1$.

Mathematical treatment of the question when the first sum is a multiple of the second, § 155.

§ 155. In the general system

$$\begin{array}{l|lllll|l} n_1 & \alpha_1\ ; & \lambda: & \beta_1\ ; & \mu & & \sigma_1 \\ n_2 & \alpha_2\ ; & ,, : & \beta_2\ ; & ,, & & \sigma_2, \end{array}$$

in which we suppose that the first sum is k times the second, *i.e.* $\sigma_1 = k\sigma_2$, we find, by proceeding as in § 144, that

$$\lambda : \mu :: n_1 - \alpha_1 - k\beta_2 : kn_2 - \alpha_1 - k\beta_2,$$

and, by proceeding as in § 147, that

$$\lambda : \mu :: \beta_1 - k\beta_2 : kn_2 - n_1 + \beta_1 - k\beta_2.$$

We thus have the two systems

$$\begin{array}{l|lllll|l} n_1 & \alpha_1\ ; & n_1 - r : & \beta_1\ ; & kn_2 - r & & k\sigma \\ n_2 & \alpha_2\ ; & ,, \quad : & \beta_2\ ; & ,, & & \sigma \end{array} \ldots\ldots\ldots\text{(i)},$$

where $r = \alpha_1 + k\beta_2$; and

$$\begin{array}{c|lllll|l} n_1 & \alpha_1\,; & s: & \beta_1\,; & kn_2 - n_1 + s & & k\sigma \\ n_2 & \alpha_2\,; & ,,: & \beta_2\,; & ,, & & \sigma \end{array} \quad \ldots\ldots\ldots \text{(ii)},$$

where $s = \beta_1 - k\beta_2$.

We may also express the system more symmetrically in the form

$$\begin{array}{c|llll|l} n_1 & \alpha_1: & \beta_1 - k\beta_2: & \beta_1\,; & k\alpha_2 - \alpha_1 & k\sigma \\ n_2 & \alpha_2\,; & ,, \quad : & \beta_2\,; & ,, & \sigma \end{array} \quad \ldots \text{(iii)}.$$

In (i) r may have the values 0, 1, 2, ..., $n_1 - 1$; and α_1 and β_2 are any numbers such that $\alpha_1 + k\beta_2 = r$. If zero values are excluded, the values of r are $k+1$, $k+2$, ..., $n_1 - 1$.

In (ii) β_1 must be greater than $k\beta_2$, and s may have the values 1, 2, 3, ..., n_1. If zero values are excluded, the values of s are 1, 2, 3, ..., $n_1 - k_1 - 1$.

The case in which the first sum is twice the second, §§ 156–159.

§ 156. Taking Leonardo's case of $n_1 = 12$, $n_2 = 32$ and putting $k = 2$, we see that

$$\frac{\mu}{\lambda} = \frac{64 - r_2}{12 - r_2},$$

where $r_2 = \alpha_1 + 2\beta_2$ and has the values 0, 1, 2, ..., 11, or

$$\frac{\mu}{\lambda} = \frac{52}{s_2} + 1,$$

where $s_2 = \beta_1 - 2\beta_2$ and has the values 1, 2, 3, ..., 12.

Taking the latter form, we find that corresponding to the values

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12$$

of s_2, the values of $\frac{\mu}{\lambda}$ expressed in their lowest terms are

$$\tfrac{53}{1},\ \tfrac{27}{1},\ \tfrac{55}{3},\ \tfrac{14}{1},\ \tfrac{57}{5},\ \tfrac{29}{3},\ \tfrac{59}{7},\ \tfrac{15}{2},\ \tfrac{61}{9},\ \tfrac{31}{5},\ \tfrac{63}{11},\ \tfrac{16}{3}.$$

The systems therefore are

$$s_2 = 1 \quad \begin{array}{r|l|l|l|l|l|l} 12 & 1;1:11;53 & 584, & 3;1:\ 9;53 & 480, \ldots, & 11;1:1;53 & 64 \\ 32 & 27;,,:\ 5;,, & 292 & 28;,,:\ 4;,, & 240 & 32;,,:0;\ ,, & 32, \end{array}$$

$$s_2 = 2 \quad \begin{array}{r|l|l|l|l|l|l} 12 & 0;1:12;27 & 324, & 2;1:10;27 & 272, \ldots, & 10;1:2;27 & 64 \\ 32 & 27;,,:\ 5;,, & 162 & 28;,,:\ 4;,, & 136 & 32;,,:0;,, & 32, \end{array}$$

$$s_2 = 3 \quad \begin{array}{r|l|l|l|l|l|l} 12 & 1;3:11;55 & 608, & 3;3:\ 9;55 & 504, \ldots, & 9;3:3;55 & 192 \\ 32 & 28;,,:\ 4;,, & 304 & 29;,,:\ 3;,, & 252 & 32;,,:0;,, & 96, \end{array}$$

$s_2 = 4$	12	0;1:12;14	168,	2;1:10;14	142,...,	8;1:4;14	64
	32	28;,,: 4;,,	84	29;,,: 3;,,	71	32;,,:0;,,	32,
$s_2 = 5$	12	1;5:11;57	632,	3;5: 9;57	528,...,	7;5:5;57	320
	32	29;,,: 3;,,	316	30;,,: 2;,,	264	32;,,:0;,,	160,
$s_2 = 6$	12	0;3:12;29	348,	2;3:10;29	296,...,	6;3:6;29	192
	32	29;,,: 3;,,	174	30;,,: 2;,,	148	32;,,:0;,,	96,
$s_2 = 7$	12	1;7:11;59	656,	3;7: 9;59	552,...,	5;7:7;59	448
	32	30;,,: 2;,,	328	31;,,: 1;,,	276	32;,,:0;,,	224,
$s_2 = 8$	12	0;2:12;15	180,	2;2:10;15	154,	4;2:8;15	128
	32	30;,,: 2;,,	90	31;,,: 1:,,	77	32;,,:0;,,	64,
$s_2 = 9$	12	1;9:11;61	680,	3;9: 9;61	576		
	32	31;,,: 1;,,	340	32;,,: 0;,,	288,		
$s_2 = 10$	12	0;5:12;31	372,	2;5:10;31	320		
	32	31;,,: 1;,,	186	32;,,: 0:,,	160,		
$s_2 = 11$	12	1;11:11;63	704				
	32	32; ,,: 0;,,	352.				
$s_2 = 12$	12	0; 3:12;16	192				
	32	32; ,,: 0;,,	96.				

§ 157. The example given by Leonardo (§ 138) is

12	6 ;	1 :	6 ;	14	90
32	31 ;	,, :	1 ;	,,	45,

which in the above series of systems is the fourth in the set for $s_2 = 4$. Leonardo derived his system from the equal-sum system

12	6 ;	1 :	6 ;	5	36
32	31 ;	,, :	1 ;	,,	,, ,

and it is evident that to every system in which the first sum is double of the second there corresponds an equal-sum system in which β_1, β_2 are the same as in the first system (and therefore also α_1, α_2), but not of course λ, μ; for if $\beta_1 \gtreqless 2\beta_2$ it follows that $\beta_1 \gtreqless \beta_2$; and conversely to every equal-sum

system for which $\beta_1 \gtreqless 2\beta_2$ there corresponds a system with the same values of β_1, β_2 in which the first sum is double of the second.

§ 158. In the case of equal sums, when $n_1 = 12$ and $n_2 = 32$, the values of $\frac{\mu}{\lambda}$ are given by $\frac{30}{s_1} + 1$, where $s_1 = \beta_1 - \beta_2$ and has the values 1, 2, 3, ..., 12. Thus, corresponding to the values

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12$$

of s_1, the values of $\frac{\mu}{\lambda}$, expressed in their lowest terms, are

$$\tfrac{21}{1},\ \tfrac{11}{1},\ \tfrac{23}{3},\ \tfrac{6}{1},\ \tfrac{5}{1},\ \tfrac{13}{3},\ \tfrac{27}{7},\ \tfrac{7}{2},\ \tfrac{29}{9},\ \tfrac{3}{1},\ \tfrac{31}{11},\ \tfrac{8}{3}.$$

The following table shows for all systems in which $n_1 = 12$, $n_2 = 32$ all the possible values of β_1, β_2 for which a system exists with the first sum double of the second and the values of λ, μ corresponding to this system, and also the values of λ, μ corresponding to the same values of β_1, β_2 in which the sums are equal.

		1,21	1,11	3,23	1,6	1,5	3,13	7,27	2,7	9,29	1,3	11,31	3,8
		1	2	3	4	5	6	7	8	9	10	11	12
1, 53	1	$\genfrac{}{}{0pt}{}{1}{0}$	$\genfrac{}{}{0pt}{}{3}{1}$	$\genfrac{}{}{0pt}{}{5}{2}$	$\genfrac{}{}{0pt}{}{7}{3}$	$\genfrac{}{}{0pt}{}{9}{4}$	$\genfrac{}{}{0pt}{}{11}{5}$						
1, 27	2		$\genfrac{}{}{0pt}{}{2}{0}$	$\genfrac{}{}{0pt}{}{4}{1}$	$\genfrac{}{}{0pt}{}{6}{2}$	$\genfrac{}{}{0pt}{}{8}{3}$	$\genfrac{}{}{0pt}{}{10}{4}$	$\genfrac{}{}{0pt}{}{12}{5}$					
3, 55	3			$\genfrac{}{}{0pt}{}{3}{0}$	$\genfrac{}{}{0pt}{}{5}{1}$	$\genfrac{}{}{0pt}{}{7}{2}$	$\genfrac{}{}{0pt}{}{9}{3}$	$\genfrac{}{}{0pt}{}{11}{4}$					
1, 14	4				$\genfrac{}{}{0pt}{}{4}{0}$	$\genfrac{}{}{0pt}{}{6}{1}$	$\genfrac{}{}{0pt}{}{8}{2}$	$\genfrac{}{}{0pt}{}{10}{3}$	$\genfrac{}{}{0pt}{}{12}{4}$				
5, 57	5					$\genfrac{}{}{0pt}{}{5}{0}$	$\genfrac{}{}{0pt}{}{7}{1}$	$\genfrac{}{}{0pt}{}{9}{2}$	$\genfrac{}{}{0pt}{}{11}{3}$				
3, 29	6						$\genfrac{}{}{0pt}{}{6}{0}$	$\genfrac{}{}{0pt}{}{8}{1}$	$\genfrac{}{}{0pt}{}{10}{2}$	$\genfrac{}{}{0pt}{}{12}{3}$			
7, 59	7							$\genfrac{}{}{0pt}{}{7}{0}$	$\genfrac{}{}{0pt}{}{9}{1}$	$\genfrac{}{}{0pt}{}{11}{2}$			
2, 15	8								$\genfrac{}{}{0pt}{}{8}{0}$	$\genfrac{}{}{0pt}{}{10}{1}$	$\genfrac{}{}{0pt}{}{12}{2}$		
9, 61	9									$\genfrac{}{}{0pt}{}{9}{0}$	$\genfrac{}{}{0pt}{}{11}{1}$		
5, 31	10										$\genfrac{}{}{0pt}{}{10}{0}$	$\genfrac{}{}{0pt}{}{12}{1}$	
11, 63	11											$\genfrac{}{}{0pt}{}{11}{0}$	
3, 16	12												$\genfrac{}{}{0pt}{}{12}{0}$

The values of β_1, β_2, placed one above the other, are shown in the body of the table (and from them the values of α_1, α_2 can be derived, since $\alpha_1 = 12 - \beta_1$, $\alpha_2 = 32 - \beta_2$). The number above the column in which β_1, β_2 occurs is the value of s_1, $= \beta_1 - \beta_2$, and the pair of numbers above are the values of λ, μ for this value of s_1, *i.e.* when the sums are equal. The number in the column to the left of β_1, β_2 is the value of s_2, $= \beta_1 - 2\beta_2$, and the pair of numbers to its left are the values of λ, μ for this value of s_2, *i.e.* when the first sum is double of the second.

For example, taking $\beta_1, \beta_2 = 8, 2$, the two systems are

4 ; 3 : 8 ; 13	116 and	4 ; 1 : 8 ; 14	116
30 ; ,, : 2 ; ,,	,,	30 ; ,, : 2 ; ,,	58,

3, 13 being at the top of the column in which 8, 2 occurs and 1, 14 at the side, and the values of s_1 and s_2 being 6 and 4.

§ 159. Leonardo clearly intended that in all his systems one of the moduli should be unity and the other an integer, and if in both the equal-sum system and the derived system this condition is to be satisfied, the only possible values or β_1, β_2 are contained in the following table, which is extracted from the large table just given.

		1, 21	1, 11	1, 6	1, 5
		1	2	4	5
1, 53	1	$\genfrac{}{}{0pt}{}{1}{0}$	$\genfrac{}{}{0pt}{}{3}{1}$	$\genfrac{}{}{0pt}{}{7}{3}$	$\genfrac{}{}{0pt}{}{9}{4}$
1, 27	2		$\genfrac{}{}{0pt}{}{2}{0}$	$\genfrac{}{}{0pt}{}{6}{2}$	$\genfrac{}{}{0pt}{}{8}{3}$
1, 14	4			$\genfrac{}{}{0pt}{}{4}{0}$	$\genfrac{}{}{0pt}{}{6}{1}$

Thus there are only nine cases of this kind out of 42, and if zero values are excluded only six out of 25.

The case taken by Leonardo is that $\beta_1, \beta_2 = 6, 1$, the last in this table.

The case in which the first sum is three times the second, §§ 160–163.

§ 160. Proceeding as in § 155 and putting $k = 3$ we see that

$$\frac{\mu}{\lambda} = \frac{96 - r_3}{12 - r_3},$$

where $r_3 = \alpha_1 + 3\beta_2$ and has the values 0, 1 2 ..., 11; or that

$$\frac{\mu}{\lambda} = \frac{84}{s_3} + 1,$$

where $s_3 = \beta_1 - 3\beta_2$ and has the values 1, 2, 3, ..., 12.

Taking the latter form we find that, corresponding to the values

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12$$

of s_3, the values of $\frac{\mu}{\lambda}$, expressed in their lowest terms, are

$$\frac{85}{1},\ \frac{43}{1},\ \frac{29}{1},\ \frac{22}{1},\ \frac{89}{5},\ \frac{15}{1},\ \frac{13}{1},\ \frac{23}{2},\ \frac{31}{3},\ \frac{47}{5},\ \frac{95}{11},\ \frac{8}{1}.$$

§ 161. The following table which corresponds to that in § 158 gives, for all admissible values of β_1, β_2, the values of λ, μ for systems in which the sums are equal and for those in which the first sum is three times the second.

		1,21	1,11	3,23	1,6	1,5	3,13	7,27	2,7	9,29	1,3	11,31	3,8
		1	2	3	4	5	6	7	8	9	10	11	12
1, 85	1	$\genfrac{}{}{0pt}{}{1}{0}$		$\genfrac{}{}{0pt}{}{4}{1}$		$\genfrac{}{}{0pt}{}{7}{2}$		$\genfrac{}{}{0pt}{}{10}{3}$					
1, 43	2		$\genfrac{}{}{0pt}{}{2}{0}$		$\genfrac{}{}{0pt}{}{5}{1}$		$\genfrac{}{}{0pt}{}{8}{2}$		$\genfrac{}{}{0pt}{}{11}{3}$				
1, 29	3			$\genfrac{}{}{0pt}{}{3}{0}$		$\genfrac{}{}{0pt}{}{6}{1}$		$\genfrac{}{}{0pt}{}{9}{2}$		$\genfrac{}{}{0pt}{}{12}{3}$			
1, 22	4				$\genfrac{}{}{0pt}{}{4}{0}$		$\genfrac{}{}{0pt}{}{7}{1}$		$\genfrac{}{}{0pt}{}{10}{2}$				
5, 89	5					$\genfrac{}{}{0pt}{}{5}{0}$		$\genfrac{}{}{0pt}{}{8}{1}$		$\genfrac{}{}{0pt}{}{11}{2}$			
1, 15	6						$\genfrac{}{}{0pt}{}{6}{0}$		$\genfrac{}{}{0pt}{}{9}{1}$		$\genfrac{}{}{0pt}{}{12}{2}$		
1, 13	7							$\genfrac{}{}{0pt}{}{7}{0}$		$\genfrac{}{}{0pt}{}{10}{1}$			
2, 23	8								$\genfrac{}{}{0pt}{}{8}{0}$		$\genfrac{}{}{0pt}{}{11}{1}$		
3, 31	9									$\genfrac{}{}{0pt}{}{9}{0}$		$\genfrac{}{}{0pt}{}{12}{1}$	
5, 47	10										$\genfrac{}{}{0pt}{}{10}{0}$		
11,95	11											$\genfrac{}{}{0pt}{}{11}{0}$	
1, 8	12												$\genfrac{}{}{0pt}{}{12}{0}$

The numerals in the row at the top are the values of $s_1, = \beta_1 - \beta_2$, as before, and above are the values of λ, μ for equal-sum systems; and in the columns of numerals to the left are the values of $s_3, = \beta_1 - 3\beta_2$; and to the left are the values of λ, μ for the systems in which the first sum is three times the second.

As an example, take $\beta_1, \beta_2 = 6, 1$ and the table shows that the two systems are

6;	1:	6;	5	36	and	6;	1:	6;	29	180
31;	,,:	1;	,,	,,		31;	,,:	1;	,,	60.

This is Leonardo's example.

§ 162. The following table gives the values of β_1, β_2 for which both pairs of moduli are of the form $1, \mu$.

		1, 21	1, 11	1, 6	1, 5	1, 3
		1	2	4	5	10
1, 85	1	1 0			7 2	
1, 43	2		2 0	5 1		
1, 29	3				6 1	
1, 22	4			4 0		
1, 15	6					12 2

Thus there are only seven cases of this kind out of 30, and if zero values are excluded, only three out of 15.

§ 163. Denoting the first sum in a system by σ_1, and the second sum by σ_2, the only case in which the values of β_1, β_2 are common to the three sets of systems in which $\sigma_1 = \sigma_2$, $\sigma_1 = 2\sigma_2$, and $\sigma_1 = 3\sigma_2$ is, if we exclude the zero systems, that of 6, 1 which is the example given by Leonardo. This would indicate that he must have made a detailed examination of the systems.

It is singular that he should have deduced his systems for $\sigma_1 = 2\sigma_2$ and $\sigma_1 = 3\sigma_2$ from a system for $\sigma_1 = \sigma_2$ instead of finding the systems by a direct process similar to that which he had used in the case of $\sigma_1 = \sigma_2$ (§§ 116 and 144). For an exactly similar procedure would have given him the values of r, where r denoted the sum of the apples sold by the first

man in the first market and twice the number sold by the second man in the second market, or where it denoted the sum of the apples sold by the first man in the first market and three times the number sold by the second man in the second market, &c., as was shown in §§ 156 and 160.

Mathematical treatment of the question when the second sum is a multiple of the first, § 164.

§ 164. By proceeding as in § 155 we find that if $\sigma_2 = k\sigma_1$, we have the two systems

$$\begin{array}{c|ccccc|c} n_1 & \alpha_1; & kn_1 - r: & \beta_1; & n_2 - r & & \sigma \\ n_2 & \alpha_2; & ,, \quad : & \beta_2; & ,, & & k\sigma \end{array} \;\ldots\ldots\ldots(\text{i}),$$

where $r = k\alpha_1 + \beta_2$; and

$$\begin{array}{c|ccccc|c} n_1 & \alpha_1; & s: & \beta_1; & n_2 - kn_1 + s & & \sigma \\ n_2 & \alpha_2; & ,,: & \beta_2; & ,, & & k\sigma \end{array} \;\ldots\ldots\ldots(\text{ii}),$$

where $s = k\beta_1 - \beta_2$; and there is also the symmetrical system

$$\begin{array}{c|ccccc|c} n_1 & \alpha_1; & k\beta_1 - \beta_2: & \beta_1; & \alpha_2 - k\alpha_1 & & \sigma \\ n_2 & \alpha_2; & ,, \quad : & \beta_2; & ,, & & k\sigma \end{array} \;\ldots.(\text{iii}).$$

In (i) r may have any value less than kn_1 or n_2, whichever of the two is smaller. In (ii) s must be less than kn_1, and if kn_1 is greater than n_2, then s must also be $> kn_1 - n_2$.

The case in which the second sum is four times the first, §§ 165—167.

§ 165. Taking Leonardo's case in which $n_1 = 12$, $n_2 = 12$, $k = 4$, these formulæ show that

$$\frac{\mu}{\lambda} = \frac{32 - r}{48 - r},$$

where $r = 4\alpha_1 + \beta_2$ and has the values 0, 1, 2, 3, ..., 31; or that

$$\frac{\mu}{\lambda} = 1 - \frac{16}{s},$$

where s has the values 17, 18, 12, ..., 48.

Taking the latter form we see that corresponding to the values 17, 18, 19, 20, ..., 48 of s, the values of $\frac{\mu}{\lambda}$, expressed

in their lowest terms, are

$$\tfrac{1}{17},\ \tfrac{1}{9},\ \tfrac{3}{19},\ \tfrac{1}{5},\ \tfrac{5}{21},\ \tfrac{3}{11},\ \tfrac{7}{23},\ \tfrac{1}{3},\ \tfrac{9}{25},\ \tfrac{5}{13},\ \tfrac{11}{27},\ \tfrac{3}{7},\ \tfrac{13}{29},\ \tfrac{7}{15},\ \tfrac{15}{31},\ \tfrac{1}{2},$$

$$\tfrac{17}{33},\ \tfrac{9}{17},\ \tfrac{19}{35},\ \tfrac{5}{9},\ \tfrac{21}{37},\ \tfrac{11}{19},\ \tfrac{23}{39},\ \tfrac{3}{5},\ \tfrac{25}{41},\ \tfrac{13}{21},\ \tfrac{27}{43},\ \tfrac{7}{11},\ \tfrac{29}{45},\ \tfrac{15}{23},\ \tfrac{31}{47},\ \tfrac{2}{3},$$

and we note that the first modulus is greater than the second.

We may thus form the following table in which the values of β_2 are placed in the same column under the value of β_1 which is above the column. The numerals to the left of the column are the values of s, $=4\beta_1-\beta_2$, which is the same for all the values of β_2 in the same line; and to the left of any value of s are the corresponding values of λ, μ.

	$\beta_1 =$	5	6	7	8	9	10	11	12
17, 1	17	3	7	11	15	19	23	27	31
9, 1	18	2	6	10	14	18	22	26	30
19, 3	19	1	5	9	13	17	21	25	29
5, 1	20	0	4	8	12	16	20	24	28
21, 5	21		3	7	11	15	19	23	27
11, 3	22		2	6	10	14	18	22	26
23, 7	23		1	5	9	13	17	21	25
3, 1	24		0	4	8	12	16	20	24
25, 9	25			3	7	11	15	19	23
......	...	......	......	......	......	......	......	......	......
47, 31	47								1
3, 2	48								0

In each column the numbers representing β_2 are to be continued till zero is reached. As an example of the manner of using the table, take β_1, $\beta_2=7$, 10: here the value of s, $=4\beta_1-\beta_2$, is 18, and λ, μ are 9, 1: thus the system is

12	5;	9:	7;	1	52
32	22;	,,:	10;	,,	208.

Leonardo's example (§ 141) is that of β_1, $\beta_2=6$, 4.

§ 166. In the cases where $\beta_1>\beta_2$ the systems in which $\sigma_2=4\sigma_1$ can be derived from a system in which $\sigma_2=\sigma_1$, as in Leonardo's example of β_1, $\beta_2=6$, 4; but these cases form only a small proportion of the number of systems in which $\sigma_2=4\sigma_1$. Thus the total number of systems for which $\sigma_2=4\sigma_1$ is 144, but the total number of equal-sum systems from which they are derivable (*i.e.* in which β_1, β_2 are the same in both systems) is only 67. If zero values are excluded these numbers are 105 and 48.

§ 167. Leonardo's object seems to have been to transform an equal-sum system in which the moduli were 1, μ into a system with $\sigma_2 = 4\sigma_1$ in which the moduli are λ, 1. The only values of β_1, β_2 for which such a transformation is possible are 5, 3; 5, 0; 6, 4; 9, 4; 10, 8; the equal-sum and the (σ, 4σ) systems being

12	7;	1:	5;	11	62,	12	7;	17:	5;	1	124
32	29;	,,:	3;	,,	,,	32	29;	,,:	3;	,,	496,
12	7;	1:	5;	5	32,	12	7;	5:	5;	1	40
32	32;	,,:	0;	,,	,,	32	32;	,,:	0;	,,	160,
12	6;	1:	6;	11	72,	12	6;	5:	6;	1	36
32	28;	,,:	4;	,,	,,	32	28;	,,:	4;	,,	144,
12	3;	1:	9;	5	48,	12	3;	2:	9;	1	15
32	28;	,,:	4;	,,	,,	32	28;	,,:	4;	,,	60,
12	2;	1:	10;	11	112,	12	2;	2:	10;	1	14
32	24;	,,:	8;	,,	,,	32	24;	,,:	8;	,,	56,

the third transformation being that given by Leonardo.

Systems of more than two lines in which the sums are in arithmetical progression, §§ 168–169.

§ 168. Many of the systems derived from the table in § 165 may be so continued as to extend to more than two lines, the numbers represented having a common difference, and also the β's, in which case also the sums will have a common difference (§ 169).

Thus the 2-line example in § 165 may be continued indefinitely by adding 17 in the first column and 3 in the third, the numbers increasing by 20 and the sums by 156, viz. we have

12	5;	9:	7;	1	52
32	22;	,,:	10;	,,	208
52	39;	,,:	13;	,,	364
72	56;	,,:	16;	,,	520
...	...	...	...	...	... .

If $\beta_2 > \beta_1$, the system can be continued indefinitely. If $\beta_2 < \beta_1$ it can be continued with decreasing β's so long as

they remain positive. Thus the last system in § 167 can be continued by subtracting 2 in the third column and adding 22 in the first column, viz. we have

12	2 ;	2 :	10 ;	1	14
32	24 ;	,, :	8 ;	,,	56
52	46 ;	,, :	6 ;	,,	98
72	68 ;	,, :	4 ;	,,	140
92	90 ;	,, :	2 ;	,,	182
112	112 ;	,, :	0 ;	,,	224,

and the third system, which is Leonardo's own example (§ 141) viz.

12	6 ;	5 :	6 ;	1	36
32	28 ;	,, :	4 ;	,,	144,

can be continued by the line

52	50 ;	5 :	2 :	1	252.

The first system can also be continued by one line. The second and fourth systems cannot be extended.

If $\beta_2 < \beta_1$ and $2\beta_2 \geq \beta_1$ there will be at least 3 lines in the system.

§ 169. In general, if n_1, n_2 are the numbers, and σ_1, σ_2 the sums, then if σ_2 is to be equal to $k\sigma_1$ we find that λ, μ are given by the proportion

$$\lambda : \mu :: s : n_2 - kn_1 + s,$$

where $s = k\beta_1 - \beta_2$ and has the values 0, 1, 2, 3, ..., kn_1. We thus have, as in § 164, the system

$$\begin{array}{l|lllll|l} n_1 & \alpha_1\,; & s: & \beta_1\,; & n_2 - kn_1 + s & & \sigma_1 \\ n_2 & \alpha_2\,; & ,,\,; & \beta_2\,; & \quad ,, & & \sigma_2. \end{array}$$

The difference between σ_2 and σ_1 is

$$(\alpha_2 - \alpha_1)\,s - (\beta_1 - \beta_2)\,(n_2 - kn_1 + s),$$

which $$= (n_2 - n_1)\,s - (\beta_1 - \beta_2)\,(n_2 - kn_1).$$

If therefore n_3 and β_3 are such that $n_3 - n_2 = n_2 - n_1$ and $\beta_3 - \beta_2 = \beta_2 - \beta_1$, then, if σ_3 be the third sum, $\sigma_3 - \sigma_2$ will be $= \sigma_2 - \sigma_1$.

Thus, in general, if the n's and the β's (and therefore also the α's) are in arithmetical progression, so also are the σ's.

The case in which the numbers are 10, 30, 50, *and the sums are in the proportion* 1, 2, 3, § 170.

§ 170. The preceding investigations suggest that it would be of interest to determine the systems in which the numbers are 10, 30, 50 (which were those chosen by Widman and so many other writers), and the sums are in the proportion of 1, 2, 3.

To obtain these systems we take $n_1 = 10$, $n_2 = 20$, $k = 2$, and we find

$$\lambda : \mu :: s : s + 10,$$

where $s = 2\beta_1 - \beta_2$ and has the values 1, 2, 3, ..., 20.

Corresponding to these values of s the values of $\frac{\mu}{\lambda}$ in their lowest terms are

$$\tfrac{11}{1},\ \tfrac{6}{1},\ \tfrac{13}{3},\ \tfrac{7}{2},\ \tfrac{3}{1},\ \tfrac{8}{3},\ \tfrac{17}{7},\ \tfrac{9}{4},\ \tfrac{19}{9},\ \tfrac{2}{1},$$
$$\tfrac{21}{11},\ \tfrac{11}{6},\ \tfrac{23}{13},\ \tfrac{12}{7},\ \tfrac{5}{3},\ \tfrac{13}{8},\ \tfrac{27}{17},\ \tfrac{14}{9},\ \tfrac{29}{19},\ \tfrac{3}{2}.$$

We thus obtain the following table in which the numbers at the top of the table are the values of β_1, and the numbers in the column under each value of β_1 are corresponding values of β_2. The numerals 1, 2, ..., 20 in the left-hand column are the values of s, $= 2\beta_1 - \beta_2$, for each value of β_2 in the same line and of β_1 above it at the head of the column; and the pairs of numerals to the left of each value of s are the corresponding values of λ, μ.

	$\beta_1 =$	1	2	3	4	5	6	7	8	9	10
1, 11	1	1	3	5	7	9	11	13	15	17	19
1, 6	2	0	2	4	6	8	10	12	14	16	18
3, 13	3		1	3	5	7	9	11	13	15	17
2, 7	4		0	2	4	6	8	10	12	14	16
1, 3	5			1	3	5	7	9	11	13	15
3, 8	6			0	2	4	6	8	10	12	14
7, 17	7				1	3	5	7	9	11	13
4, 9	8				0	2	4	6	8	10	12
9, 19	9					1	3	5	7	9	11
1, 2	10					0	2	4	6	8	10
11, 21	11						1	3	5	7	9
6, 11	12						0	2	4	6	8
13, 23	13							1	3	5	7
7, 12	14							0	2	4	6
3, 5	15								1	3	5
8 13	16								0	2	4
17, 27	17									1	3
9, 14	18									0	2
19, 29	19										1
2, 3	20										0

Thus, for example, taking $\beta_1=7$ and $\beta_2=10$, 7, 5, we have the three systems

10	3 ; 2 : 7 ; 7	55,	10	3 ; 7 : 7 ; 17	140,	10	3 ; 9 : 7 ; 19	160
30	20 ; ,, : 10 ; ,,	110	30	23 ; ,, : 7 ; ,,	280	30	25 ; ,, : 5 ; ,,	320
50	37 ; ,, : 13 ; ,,	165	50	43 ; ,, : 7 ; ,,	420	50	47 ; ,, : 3 ; ,,	480
...	....	...	...	....	...	70	69 ; ,, : 1 ; ,,	640.

The first two systems can be continued indefinitely.

Whenever $\beta_2 >$ or $=\beta_1$ the systems can be continued indefinitely: thus there are 55 such systems. When $\beta_1 > \beta_2$, but $< 2\beta_2$, the system consists of three or more lines: the number of such systems is 25. If $\beta_1 > 2\beta_2$ there are but two lines in the system: the number of such systems is 30.

In the terminating systems the greatest number of lines is when $\beta_2 = \beta_1 - 1$ and the number of lines is then equal to β_1.

The case in which the numbers are 10, 30, 50, *and the sums are in the proportion* 3, 2, 1, § 171.

§ 171 The cases in which the sums are in the proportion of 3, 2, 1 and 5, 3, 1 are also of interest.

To obtain the systems in which n_1, n_2 are the numbers, and σ_1 is to σ_2 in the proportion p to q, we put $s = q\beta_1 - p\beta_2$, and by proceeding as before we find that

$$\lambda : \mu :: s : pn_2 - qn_1 + s,$$

where, if $qn_1 < pn_2$, s has the values 1, 2, 3, ..., qn_1.*

For $n_1 = 10$, $n_2 = 30$, $p = 3$, $q = 2$, the formula becomes

$$\lambda : \mu :: s : s + 70,$$

where $s = 2\beta_1 - 3\beta_2$ and has the values 1, 2, 3, ..., 20.

Corresponding to these values of s the values of $\frac{\mu}{\lambda}$ in their lowest terms are $\frac{71}{1}$, $\frac{36}{1}$, $\frac{73}{3}$, $\frac{37}{2}$, $\frac{15}{1}$, ..., and we thus obtain the following table in which as before the values of β_1 are at the head of the column, and the values of β_2 corresponding to β_1 are in the column under it. The numerals in the column to the left give the values of s_1, $= 2\beta_1 - 3\beta_2$, for each entry on the line, and the pair of numbers to the left are the values

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* The formula may also be written in the forms

$$\lambda : \mu :: q\beta_1 - p\beta_2 : p\alpha_2 - q\alpha_1 ;$$

and $$\lambda : \mu :: qn_1 - r : pn_2 - r,$$

where $r = q\alpha_1 + p\beta_2$.

of λ, μ corresponding to the value of s. There are no values of β_1, β_2 for which $s=19$, and this number is therefore omitted.

	$\beta_1 =$	1	2	3	4	5	6	7	8	9	10
1, 71	1		1			3			5		
1, 36	2	0			2			4			6
3, 73	3			1			3			5	
2, 37	4		0			2			4		
1, 15	5				1			3			5
3, 38	6			0			2			4	
1. 11	7					1			3		
4. 39	8				0			2			4
9, 79	9						1			3	
1, 8	10					0			2		
11, 81	11							1			3
6, 41	12						0			2	
13, 83	13								1		
1, 6	14							0			2
3, 17	15									1	
8, 43	16								0		
17, 87	17										1
9, 44	18									0	
2, 9	20										0

Unless $\beta_1 =$ or $< 2\beta_2$ there can be only two lines in the system; so that the only 3-line systems correspond to

$\beta_1, \beta_2 = 2, 1;\ 5, 3;\ 8, 5;\ 4, 2;\ 7, 4;\ 10, 6;\ 6, 3;\ 9, 5;\ 8, 4;\ 10, 5;$ and are

10	8;1: 2;71	150,	10	5;1:5;71	360,	10	2;1: 8;71	570
30	29;,,: 1; ,,	100	30	27;,,:3; ,,	240	30	25;,,: 5; ,,	380
50	50;,,: 0; ,,	50	50	49;,,:1; ,,	120	50	48;,,: 2; ,,	190,

10	6;1: 4;36	150,	10	3;1:7;36	255,	10	0;1:10;36	360
30	28;,,: 2; ,,	100	30	26;,,:4; ,,	170	30	24;,,: 6; ,,	240
50	50;,,: 0; ,,	50	50	49;,,:1; ,,	85	50	48;,,: 2; ,,	120,

10	4;3: 6;73	450,	10	1;3:9;73	660,	10	2;2: 8;37	300
30	27;,,: 3; ,,	300	30	25;,,:5; ,,	440	30	26;,,: 4; ,,	200
50	50;,,: 0; ,,	150	50	49;,,:1; ,,	220	50	50;,,: 0; ,,	100,

10	0;1:10;15	150.
30	25;,,: 5; ,,	100
50	50;,,: 0; ,,	50

Of these ten systems only four do not contain zeros.

The case in which the numbers are 10, 30, 50, *and the sums are in the proportion* 5, 3, 1, § 172.

§ 172. By putting $n_1=10$, $n_2=30$, $p=5$, $q=3$ in the general formula of the preceding section we find that σ_1 will be to σ_2 in the proportion of 5 to 3 in the systems in which

$$\lambda : \mu :: s : s+120,$$

where $s=3\beta_1-5\beta_2$ and has the values 1, 2, 3, .., 30.

Corresponding to these values of s the values of $\frac{\mu}{\lambda}$ in their lowest terms are

$\frac{121}{1}$, $\frac{61}{1}$, $\frac{41}{1}$, $\frac{31}{1}$, $\frac{25}{1}$, $\frac{21}{1}$, $\frac{127}{7}$, $\frac{16}{1}$, $\frac{129}{9}$, $\frac{13}{1}$, ..., $\frac{37}{7}$, $\frac{149}{29}$, $\frac{5}{1}$.

The following is a table similar in character to that in the preceding section, but as we are concerned only with 3-line systems (in which $\beta_1 =$ or $<2\beta_2$) only the first five lines are given, as the lower lines contain no values which satisfy the conditions:

		1	2	3	4	5	6	7	8	9	10
1, 121	1		1					4			
1, 61	2				2					5	
1, 41	3	0					3				
1, 31	4			1					4		
1, 25	5					2					5
......	...	...	...	...	...	...	...	...	...	...	...

Thus the admissible values of β_1, β_2 are 2, 1 ; 7, 4 ; 4, 2 ; 9, 5 ; 6, 3 ; 8, 4 ; 10, 5 ; and the systems are

10	8 ; 1 : 2 ; 121	250,	10	3 ; 1 : 7 ; 121	850,	10	6 ; 1 : 4 ; 61	250
30	29 ; ,, : 1 ; ,,	150	30	26 ; ,, : 4 ; ,,	510	30	28 ; ,, : 2 ; ,,	150
50	50 ; ,, : 0 ; ,,	50	50	49 ; ,, : 1 ; ,,	170	50	50 ; ,, : 0 ; ,,	50,

10	1 ; 1 : 9 ; 61	550,	10	4 ; 1 : 6 ; 41	250,	10	2 ; 1 : 8 ; 31	250
30	25 ; ,, : 5 ; ,,	330	30	27 ; ,, : 3 ; ,,	150	30	26 ; ,, : 4 ; ,,	150
50	49 ; ,, : 1 ; ,,	110	50	50 ; ,, : 0 ; ,,	50	50	50 ; ,, : 0 ; ,,	50

10	0 ; 1 : 10 ; 25	250
30	25 ; ,, : 5 ; ,,	150
50	50 ; ,, : 0 ; ,,	50.

Of these seven systems only two do not contain zeros. In all of the systems λ is unity.

Part V.

History of the puzzle-question in the seventeenth and eighteenth centuries.

Treatment of the question by Bachet (1612), §§ 173–175.

§ 173. The problem of the sale of eggs as stated in § 1 of this paper is not a satisfactory one in itself, as it involves a sort of puzzle or paradox, nor does it represent a particular use of arithmetic or method of reasoning applied to arithmetic, as is the case with several early questions which still find a place in Arithmetics*: but yet it possesses a certain amount of historical interest, if only because it attracted the attention of Widman, who not only reproduced it, but showed how similar problems might be constructed. Tartaglia also must have examined with some care the principles on which it depended, for he modified the problem to render it more precise, and free it from paradox; and he formed other problems of the same type, also including one which could be solved only in integers. The paradox in the problem consisted in the words 'at the same price', which are not justifiable, as the eggs are sold at two different prices.

Leonardo had stated the problem in a satisfactory form as a question in the partition of numbers, and his solutions were nearly as complete as was possible without the use of algebraical symbols. It would seem that after Leonardo's time it was extended to three persons and acquired its paradoxical character.

§ 174. As arithmetic and algebra advanced the problem entirely ceased to be included in books on arithmetic. It found however a place in collections of mathematical recreations and was there treated as a question in partitions and was solved in integers. Thus one of the problems in Bachet's† *Problèmes* of 1612 is: Several unequal numbers being given, to divide

* Such as the question of the cask with three taps which would empty it in different times: of the workman who was paid a certain sum for each day he worked and had to pay back a certain sum for each day he was idle: of the master who had so much over if he paid his workmen a certain sum a day and was short by so much if he paid them another sum: &c.

† "Problemes plaisans et delectables, qui se font par les nombres . . . par Clavde Gaspar Bachet . . ." (Lyons, 1612). The problem is no. xxi. on p. 106. In the edition of 1621 it is no. xxiv., p. 178.

each of them into two parts and find two multipliers for the respective parts so that the sum of the two products may be the same for all the numbers. He says that this problem is usually proposed in the form "Trois femmes vendent des pom[mes] au marché; la premiere en vend 20. la seconde 30. la troisiesme 40. Et elles vendent tout a vn mesme prix, & rapportent chascune la mesme somme d'argent, on demande comme cela se peut faire".

He remarks that, taken literally, the question is impossible, but that we must suppose that the apples are sold at different times and at different prices, though all that are sold at the same time are sold at the same price.* As an example, he takes the two prices to be one denier and three deniers, and supposes that at the first sale the numbers of apples sold are respectively 2, 17, 32, and at the second sale the numbers are 18, 13, 8: under these conditions each woman brings back 56 deniers. He points out that the solution of the question consists in dividing each of the given numbers into two parts, and finding two numbers such that multiplying one number by one part and the other by the other the sum of the products is the same; and he then states a 'general and infallible' rule which he has invented, and which, so far as he knows, no one had previously attempted†. This rule he explains by examples as follows: The numbers being 20, 30, 40 he subtracts 20 from 30 and 20 from 40, thus obtaining 10 and 20. He finds the common measures of these two numbers, viz. 2, 5, 10, and says that we can take, as multipliers, any two numbers which differ by any of these numbers, such as 1 and 3, or 1 and 6, or 1 and 11. For an example, he takes 1 and 3 and divides the difference between 40 and 20, that is 20, by the interval between the two multipliers, that is 2, giving 10: this he multiplies by 1 and 3, giving 10 and 30. He then arbitrarily divides 20, the smallest of the given numbers, into any two parts such that the larger is greater than 10, the smaller of the numbers 10 and 30. For example, he divides 20 into 3 and 17, and adds 30 to the smaller part; and subtracts 10 from the larger part, thus obtaining 33 and 7 which are the two parts into which the largest number is to

* "Il est certain que prenant cecy cruement comme il est proposé, & s' imaginant qu' elles ayent vendu toutes leurs pommes à vn seul prix, & à vne seule fois, la chose est impossible, car en ceste façon il ne peut estre que celle qui a plus grand quantité de pommes, ne rapporte d'avantage d'argent. Mais il se doit entendre, quelles vendent à diuerses fois, & à diuers prix, bien qu'a chasque fois elles vendent chascune à vn mesme prix".

† "Pour faire cecy i'ay inuenté la regle suiuante generale & infallible personne par cy deuant ne s'en estant auisé que ie sache".

divided. Similarly to find the two parts into which 30 is to be divided, he divides the difference between 30 and 20, that is 10, by the interval between 1 and 3, that is 2, the quotient being 5, which multiplied by 1 and 3 gives 5 and 15: he adds 15 to 3, the smaller part of 20, giving 18, and subtracts 5 from 17, the larger part giving 12; these numbers, 18 and 12, are the parts into which 30 is to be divided: thus when the multipliers are 1 and 3, the numbers 20, 30, 40 have been divided into the parts 3 and 17, 18 and 12, 33 and 7, and each woman brings back 54 deniers.

Expressed in the notation of the present paper the process gives the system

20	3 ;	1 :	17 ;	3	54
30	18 ;	,, :	12 ;	,,	,,
40	33 ;	,, :	7 ;	,,	,, .

§ 175. In a second example he takes the multipliers to be 2 and 7. Dividing the difference 20 by 5 he obtains 4, which multiplied by 2 and 7 gives 8 and 28. He divides 20 arbitrarily into the two parts 8 and 12, the larger being greater than 8 (the smaller of the two parts 8 and 28), and he obtains the two parts into which 40 is to be divided by adding 28 to 8 and subtracting 8 from 12, giving 36 and 4 as the two parts. To obtain the two parts into which 30 is to be divided, he divides the difference 10 by 5, giving 2, which when multiplied by 2 and 7 gives 4 and 14: he adds 14 to 8 and subtracts 4 from 12, thus obtaining 22 and 8, which are the two parts into which 30 is to be divided. The system so obtained is

20	8 ;	2 :	12 ;	7	100
30	22 ;	,, :	8 ;	,,	,,
40	36 ;	,, :	4 ;	,,	,, .

He also gives a third example in which the multipliers are 1 and 11, and the system obtained in the same manner is

20	6 ;	1 :	14 ;	11	160
30	17 ;	,, :	13 ;	,,	,,
40	28 ;	,, :	12 ;	,,	,, .

Examination of Bachet's rule, §§ 176—181.

§ 176. Although Bachet's examples all have reference to the numbers 20, 30, 40, the manner in which he explains his rule shows that it was so constructed as to apply to any three (or more) numbers. Expressed in symbols his procedure is as follows:

Let n_1, n_2, n_3 be three numbers, and let $n_2 - n_1 = d_1$, $n_3 - n_2 = d_2$. Let δ be any common divisor of d_1 and d_2, and let the quotients of d_1 and d_2 when divided by δ be δ_1 and δ_2, so that $d_1 = \delta_1 \delta$; $d_2 = \delta_2 \delta$. Take for λ and μ, the two multipliers, any numbers differing by δ, so that $\mu - \lambda = \delta$. Divide n_1 into any two parts α and β such that the larger part β is greater than $\lambda\delta_2$. Then the two parts into which n_3 is to be divided are $\alpha + \mu\delta_2$ and $\beta - \lambda\delta_2$, and the two parts into which n_2 is to be divided are $\alpha + \mu\delta_1$ and $\beta - \lambda\delta_1$.

In the notation of this paper the system therefore is

$$\begin{array}{l|lclcl|c} n_1 & \alpha & ; & \lambda : & \beta & ; \ \mu & \lambda\alpha + \mu\beta \\ n_1 + d_1 & \alpha + \mu\delta_1 & ; & ,, : & \beta - \lambda\delta_1 & ; \ ,, & ,, \\ n_1 + d_2 & \alpha + \mu\delta_2 & ; & ,, : & \beta - \lambda\delta_2 & ; \ ,, & ,, . \end{array}$$

§ 177. The only imperfection in Bachet's rule is that he divides n_1 into any two parts so that the larger part is greater than $\lambda\delta_2$, and he then makes his subtractions from the larger part and his additions to the smaller part: but the subtractions need not necessarily be made from the larger part; for if the smaller part is greater than $\lambda\delta_2$, they can be made from the smaller part and the additions made to the larger part. In fact, Bachet requires that β should be greater than α and greater than $\lambda\delta_2$, but only the latter condition is necessary; for there may be solutions in which α is greater than β.

The fact that Bachet has to determine $\lambda\delta_2$ in order to apply the condition $\beta > \lambda\delta_2$ explains why he partitions n_3 (the highest of the numbers) into its two parts $\alpha + \mu\delta_2$ and $\beta - \lambda\delta_2$ before partitioning n_2.

§ 178. It will be noticed that Bachet does not include unity among the common divisors of d_1 and d_2. From unity we obtain $\lambda = 1$, $\mu = 2$. Since $\lambda = 1$ and $\delta = 1$ we must have $\beta > \delta_2$, and, as $\beta < n_1$, it follows that $\lambda = 1$, $\mu = 2$ does not give a solution unless $n_1 > d_2 + 1$, if zero values of the α's and β's are inadmissible, as was clearly Bachet's intention. In Bachet's example $n_1 = 20$ and $d_2 = 20$, so that $n_1 = d_2$

§ 179. It will be seen that Bachet's rule extends to any number of numbers, viz. we have

$$\begin{array}{l|llllll|c}
n_1 & \alpha & ; & \lambda : & \beta & ; & \mu & \lambda\alpha+\mu\beta \\
n_1+\delta_1\delta & \alpha+\mu\delta_1 & ; & ,, : & \beta-\lambda\delta_1 & ; & ,, & ,, \\
n_1+\delta_2\delta & \alpha+\mu\delta_2 & ; & ,, : & \beta-\lambda\delta_2 & ; & ,, & ,, \\
n_1+\delta_3\delta & \alpha+\mu\delta_3 & ; & ,, : & \beta-\lambda\delta_3 & ; & ,, & ,, \\
\ldots\ldots & \ldots\ldots & & & \ldots\ldots & & & \ldots\ldots,
\end{array}$$

δ being any common divisor of all the differences between the numbers, and $\mu-\lambda$ being $=\delta$.

In this form it is evident that the system consists of selected terms from the system

$$\begin{array}{l|llllll|c}
n_1 & \alpha & ; & \lambda : & \beta & ; & \mu & \lambda\alpha+\mu\beta \\
n_1+\delta & \alpha+\mu & ; & ,, : & \beta-\lambda & ; & ,, & ,, \\
n_1+2\delta & \alpha+2\mu & ; & ,, : & \beta-2\lambda & ; & ,, & ,, \\
\ldots\ldots & \ldots\ldots & & & \ldots\ldots & & & \ldots\ldots .
\end{array}$$

§ 180. Bachet's system can also be written in the form

$$\begin{array}{l|llllll|c}
n_1 & \alpha & ; & \lambda : & \beta & ; & \mu & \lambda\alpha+\mu\beta \\
n_1+d_1 & \alpha+\dfrac{\mu d_1}{\mu-\lambda} & ; & ,, : & \beta-\dfrac{\lambda d_1}{\mu-\lambda} & ; & ,, & ,, \\
n_1+d_2 & \alpha+\dfrac{\mu d_2}{\mu-\lambda} & ; & ,, : & \beta-\dfrac{\lambda d_2}{\mu-\lambda} & ; & ,, & ,, \\
\ldots\ldots & \ldots\ldots & & & \ldots\ldots & & & \ldots\ldots,
\end{array}$$

in which μ and λ are any numbers such that $\mu-\lambda$ is a divisor of d_1, d_2, ...

§ 181. In Bachet's question the given numbers are 20, 30, 40, and proceeding as in § 68 there are ten groups of solutions, of which the first group, corresponding to $\lambda=1$, $\mu=11$, consists of the 19 systems

20	0;1:20;11	220,	20	1;1:19;11	210,...	20	18;1:2;11	40
30	11;,,:19; ,,	,,	30	12;,,:18; ,,	,,	30	29 ;,,:1; ,,	,,
40	22;,,:18; ,,	,,	40	23;,,:17; ,,	,,	40	40;,,:0; ,,	,, .

In the other nine groups I give only the first lines of the first and last systems corresponding to each value of λ, viz.

20	0 ; 2 : 20 ; 12	240,...	20	16 ; 2 : 4 ; 12	80,
20	0 ; 3 : 20 ; 13	260,...	20	14 ; 3 : 6 ; 13	120,
20	0 ; 4 : 20 ; 14	280,...	20	12 ; 4 : 8 ; 14	160,
20	0 ; 5 : 20 ; 15	300,...	20	10 ; 5 : 10 ; 15	200,
20	0 ; 6 : 20 ; 16	320,...	20	8 ; 6 : 12 ; 16	240,
20	0 ; 7 : 20 ; 17	340,...	20	6 ; 7 : 14 ; 17	280,
20	0 ; 8 : 20 ; 18	360,...	20	4 ; 8 : 16 ; 18	320,
20	0 ; 9 : 20 ; 19	380,...	20	2 ; 9 : 18 ; 19	360,
20	0 ; 10 : 20 ; 20	400.			

The first lines in the other systems of each group may be derived from the first line of the first system in the group by continually adding unity to the first number and subtracting unity from the third.

Each system may be completed by continually adding the second modulus (*i.e.* the fourth number) to the first number, and subtracting the first modulus (*i.e.* the second number) from the third number. In the last system of each group the third number is double of the second, the third column of the system being of the form 2β, β, 0.

In each group the fixed-sums diminish by 10 as the leading numbers in the system increase by 1.

Any factor common to the second and fourth columns (the two moduli) is to be divided out after the system has been constructed, the fixed-sum being divided by the same factor. The numbers of systems in the ten groups are 19, 17, 15, 13, 11, 9, 7, 5, 3, 1 respectively, making 100 in all. If we exclude those in which a zero occurs there are 81.

Two of Bachet's solutions belong to the fifth group, the common factor 5 being divided out, so that the moduli are 1 and 3; one to the fourth group, the factor 2 being divided out; and one to the first group.*

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* As recently as 1874 another edition of Bachet's work was published under the title "Problèmes plaisants & délectables . . . par Claude-Gaspar Bachet . . . Troisième édition revue, simplifiée et augmentée par A. Labosne . . ." The problem occurs on p. 122, and Bachet's rule is replaced by a somewhat vague algebraical solution. With reference to the sale in the market the editor writes (p. 124), "Le problème que nous venons de résoudre se propose ordinairement sous la forme d' une vente: Trois personnes arrivent au marché ayant respectivement 10 mesures, 12 mesures et 15 mesures de graine; elles vendent leurs marchandises aux mêmes prix, et elles en retirent chacune la même somme: on demande comment la vente a été faite".

Treatment of the question by Van Etten (1627), §§ 182—183.

§ 182. In Van Etten's *Recréation Mathématique** (1627) the problem which appears only in its concrete form of three women going to market is stated almost in Bachet's words (§ 174). Only one solution is given, viz. Bachet's first example, in which the prices at the two sales are 1 denier and 3 deniers, and the numbers of apples sold at the two sales are 2, 17, 32, and 18, 13, 8 respectively, the sum brought back being 56 deniers. Nothing is said as to how the solution was obtained, nor is it stated that there are other solutions.

In Mydorge's edition† (1630) the problem is given as in the edition of 1627, but in the notes by Denis Henrion it is mentioned that Bachet showed that there were other solutions, and he quotes the solution in which the prices are 2 and 7 deniers, and the sum brought back is 100 deniers. In the edition of 1661‡ this note is appended to the question.

§ 183. In the English translation§ (1633), it is said in the statement of the problem that 'they sold as many for a penny, the one as the other'. This suggests the earlier class of solutions in which some of the apples were sold at so many for a penny: but the misleading words are due to the translator, the statement in the original being 'elles vendent tout à vn mesme prix'. After they have sold respectively 2, 17, 32 apples at 1 denier each, the translator proceeds: "then A. said she would not sell her *apples* so cheape, but would sell them for 3 pence the peece". But in the original there is merely the statement "La seconde fois elles vendront le reste de leurs pommes 3. deniers la pomme".

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* "Recreation mathematicqve composee de plvsievrs problemes plaisants et facetievx . . ." (Lyons, 1627). The dedication is signed H. Van Etten, but the real name of the author was Jean Leurechon. The problem is no. lxix., p. 86.

† "Examen dv livre des recreations mathematiqves: . . . Par Clavde Mydorge . . ." (Paris, 1630) followed by "Nottes svr les recreations mathematiqves . . . Par D. H. P. E. M" (Paris, 1630). The problem is no. lxix., p. 127. The note occurs on p. 22 of the 'Nottes'.

‡ "Les recreations mathematiqves avec l'examen de ses problemes. . . . Premierement reveu par D. Henrion. Depuis par M. Mydorge . . . Cinquiesme & derniere Edition" (Paris, 1661). The problem and note are on pp. 144 and 145.

§ Mathematicall Recreations . . . Not vulgarly made manifest untill this time: . . . lately compiled in French, by Henry Van Etten . . . 1633." The problem is no. lxii., p. 90.

Treatment of the question by Ozanam, § 184.

§ 184. In Ozanam's Recreations* (1696) the question of the women selling apples is given as the sequel to a partition problem which is the same as Bachet's, viz. to divide several numbers each into two parts, and find two numbers, so that multiplying the first part of each of the given numbers by the first number found, and the second by the second, the sum of the two products is the same for all.

Ozanam takes as his given numbers 10, 25, 30, and he explains in detail his procedure which differs in several respects from that of Bachet. His rule is: Take for the two numbers sought (*i.e.* the multipliers) any numbers whose difference is 1 or a number which can divide exactly the product of the greater of the two numbers and the difference between any two of the three given numbers, and which are such that the greater of the numbers multiplied by the least of the given numbers, viz. 10, surpasses the smaller of the numbers multiplied by the greatest of the given numbers, viz. 30. He takes 2 and 7 as the two numbers. He then proceeds: the first part of the given number 10 may be taken at will, but must be less than 10, and less than the number obtained by subtracting 2 multiplied by the greatest number 30 from 7 multiplied by the least number 10 and dividing the remainder by 5, the difference of the numbers 7 and 2 (*i.e.* less than $\frac{7 \times 10 - 2 \times 30}{7 - 2}$).† This quotient is equal to 2, so that the first part of 10 must be 1 and the second part 9: and the sum of the products of each part and its multiplier is 65. The rule continues: to find the first part of the second given number, viz. 25, multiply the difference of the first two of the given numbers, viz. 15, by the greater number 7, and divide the product 105 by the difference between 7 and 2, viz. 5, thus giving 21, which added to 1, the first part of the first of

* "Recreations mathematiques et physiques . . . Par Mr. Ozanam" (Amsterdam, 1696). The problem is no. xxiv., p. 77, and the appended question on the sale of the apples occurs on pp. 79–80. According to Poggendorff the first edition was published at Paris in 1694 in two volumes. An English translation was published at London in 1708 under the title "Recreations mathematical and physical . . . By Monsieur Ozanam . . . Done into English . . ". The problem and question occupy pp 68–70. Denier is translated farthing. In referring to editions of Ozanam and other writers I confine myself to those which I have seen.

† The rather complicated sentence in which this condition is expressed in the edition of 1696 is in the edition of 1723 and later editions divided into two sentences, the first stating that the part must be less than 10 and than $7 \times 10 - 2 \times 30$, and the second that it must be less than the quotient of the latter number divided by $7-2$.

the given numbers gives 22, which is the first part of the second given number 25, the second part therefore being 3.

To find the first part of the third number 30, multiply the difference 5 between 25 and 30, the second and third of the given numbers, by 7, giving 35, and divide by 5, the difference between 7 and 2, the quotient being 7, which, added to 22, the first part of the second given number, gives 29, which is the first part of the third given number, the other part being 1.

He then points out that we might multiply 20, the difference of the first and third of the given numbers, by 7, and divide the product 140 by 5, giving 28, which, added to 1, the first part of the first given number, gives 29 for the first part of the third given number.

He then takes 1 and 6 for the multipliers and 4 for the first part of the given number 10, the other part being then 6 and the sum of the products 40: he merely states that the parts of the second given number are 22 and 3, and of the third 28 and 2, and verifies that the sum of the products is the same.

Comparison of Ozanam's and Bachet's processes, §§ 185–186.

§ 185. Ozanam's process may be described algebraically as follows: Let n_1, n_2, n_3 be the three given numbers, and let $n_2 = n_1 + d$, $n_3 = n_2 + d'$: take λ and μ to be any two numbers such that $\mu - \lambda$ is a divisor of μd and $\mu d'$, and that $\mu n_1 > \lambda n_3$. Take α_1, the first part of n_1, to be any number $< n_1$ and $< \dfrac{\mu n_1 - \lambda n_3}{\mu - \lambda}$; then the first part α_2 of n_2 is obtained by adding $\dfrac{\mu d}{\mu - \lambda}$ to α_1, and the first part α_3 of n_3 by adding $\dfrac{\mu d'}{\mu - \lambda}$ to α_2, or, proceeding directly from α_1 to α_3, by adding $\dfrac{\mu d_2}{\mu - \lambda}$ to α_1, where $d_2 = n_3 - n_1$.

§ 186. The main difference between Ozanam's and Bachet's procedure is that Ozanam requires that α_1 should be less than $\dfrac{\mu n_1 - \lambda n_3}{\mu - \lambda}$. This expression is equal to

$$n_1 - \frac{\lambda (n_3 - n_1)}{\mu - \lambda},$$

so that the condition is equivalent to requiring that β_1 should be greater than $\dfrac{\lambda d_2}{\mu - \lambda}$, which is Bachet's condition.

Other minor points of difference are that Ozanam requires that μd and $\mu d'$ should be divisible by $\mu - \lambda$ instead of the simpler condition that d and d' should be so divisible; but as μd and $\mu d'$ have to be calculated in order to obtain α_2 and α_3, Ozanam probably thought it convenient to calculate them at once and express the condition by their means. Also, as α_1 is a part of n_1, it was unnecessary to state that it must be less than n_1.

Ozanam's partition problem is followed by the same question expressed in the form of the three women selling apples in the market, in which Ozanam retains the paradox of the question by saying that they sell 'au même prix'. The solutions he gives are the same as those which he has found in the partition question.

Ozanam's algebraical solution, §§ 187–189.

§ 187. In the edition of the *Recreations*, published in 1723,* the partition problem and the question of the women selling apples are reproduced nearly as in that of 1696: but at the end, under the heading "Remarques", Ozanam gives an algebraical solution of the question from which, expressed in the notation of the present paper, he derives the formulæ

$$\beta_1 = \beta_2 + \frac{15\lambda}{\mu - \lambda},\quad \beta_1 = \beta_3 + \frac{20\lambda}{\mu - \lambda},\quad \beta_2 = \beta_3 + \frac{5\lambda}{\mu - \lambda},$$

which show that $\mu - \lambda$ must be a divisor of 15, 20, and 5. Thus the only admissible values of λ and μ are those for which $\mu - \lambda = 5$.

Putting $\lambda = 1$, $\mu = 6$, the last two equations become $\beta_2 = \beta_3 + 1$, $\beta_1 = \beta_3 + 4$, whence putting $\beta_3 = 0, 1, 2, 3, 4, 5, 6$, the corresponding pairs of values of β_2, β_1 are (1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9), (7, 10); putting $\lambda = 2$, $\mu = 7$, the formulæ are $\beta_2 = \beta_3 + 2$, $\beta_1 = \beta_3 + 8$, whence putting $\beta_3 = 0, 1, 2$, the corresponding values of β_2, β_1 are (2, 8), (3, 9), (4, 19); ten systems being so obtained.

Ozanam does not mention any more solutions, and it is clear that there are no more, for if $\lambda = 3$, $\mu = 8$ we have $\beta_1 = \beta_3 + 12$, so no value of β_3 gives a possible value of β_1: and similarly for higher values of λ.

The β's are the numbers of apples sold at the higher price, and it is interesting to note that in working out the question

* "Recreations mathematiques et physiques . . . Par feu M. Ozanam . . . nouvelle edition, Revûe, corrigée, & augmentée, . . ." (Paris, 1723). In 4 volumes. The problem is no. xxviii., and the problem and question occupy pp. 199–210 of vol. i. They occupy the same pages in the editions of 1741, 1750, and 1770.

algebraically Ozanam calculates the β's, taking β_3 as his fundamental quantity, although the rules were directed to the calculation of the α's. But in this Ozanam showed his grasp of the subject, for β_3 is the quantity that is in danger of becoming negative unless β_1 and therefore α_1 is properly chosen: so that by giving to β_3 the values 0, 1, 2, 3, ... he ensured that this condition was fulfilled.

Ozanam mentions (p. 203) that the question has been treated 'dans la seconde partie de l'Arithmétique universelle, p. 456', and that only six solutions are there given, the largest number of deniers brought back being 65. I have not seen the *Arithmétique Universelle*,* but it would seem likely that the author had excluded the four solutions containing zero in accordance with the practice of preceding writers. So far as I know Ozanam was the first to take account of these zero solutions, and it would thus appear that they were first included when the solutions were obtained by algebra.

§ 188. Proceeding as in § 187 and giving λ the values 1, 2, ... so that μ has the values 6, 7, ... and writing down the complete systems in which the numbers are 10, 15, 20, 25, 30 (which include the numbers 10, 25, 30) we have the group of seven systems

10	0 ; 1 : 10 ; 6	60,	10	1 ; 1 : 9 ; 6	55, ...,	10	6 ; 1 : 4 ; 6	30
15	6 ; ,, : 9 ; ,,	,,	15	7 ; ,, : 8 ; ,,	,,	15	12 ; ,, : 3 ; ,,	,,
20	12 ; ,, : 8 ; ,,	,,	20	13 ; ,, : 7 ; ,,	,,	20	18 ; ,, : 2 ; ,,	,,
25	18 ; ,, : 7 ; ,,	,,	25	19 ; ,, : 6 ; ,,	,,	25	24 ; ,, : 1 ; ,,	,,
30	24 ; ,, : 6 ; ,,	,,	30	25 ; ,, : 5 ; ,,	,,	30	30 ; ,, : 0 ; ,,	,, ,

and for $\lambda = 2$ there is a group of three systems

10	0 ; 2 : 10 ; 7	70,	10	1 ; 2 : 9 ; 7	65,	10	2 ; 2 : 8 ; 7	60
15	7 ; ,, : 8 ; ,,	,,	15	8 ; ,, : 7 ; ,,	,,	15	9 ; ,, : 6 ; ,,	,,
20	14 ; ,, : 6 ; ,,	,,	20	15 ; ,, : 5 ; ,,	,,	20	16 ; ,, : 4 ; ,,	,,
25	21 ; ,, : 4 ; ,,	,,	25	22 ; ,, : 3 ; ,,	,,	25	23 ; ,, : 2 ; ,,	,,
30	28 ; ,, : 2 ; ,,	,,	30	29 ; ,, : 1 ; ,,	,,	30	30 ; ,, : 0 ; ,,	,, .

* In Montucla's editions of Ozanam's *Recreations* (1790) the author is stated to be de Lagny, the sentence as there given being "On lit dans la seconde partie de l'Arithmétique universelle de M. de Lagny, p. 456, que cette question n'a que six solutions; en quoi cet auteur s'est trompé, car nous venons d'en indiquer 10".

For $\lambda = 3, 4, 5$ there are are no six-line systems.

There are ten systems in all, or six if we exclude those involving zeros. These or rather the systems derived from them by selecting the first and last two lines are those given in the 1723 edition of Ozanam.

§ 189. In the edition of Ozanam's *Recreations** edited by Montucla, the problem on partitions is omitted, only the question about the sales in the market being retained.† Montucla does not give Ozanam's rule, but he states that the difference of the prices must be a divisor of 15, 20, 5 and is therefore equal to 5, and he gives the ten solutions. He then, under the heading 'Remarques' gives Ozanam's algebraical solution and shows that there could not be more than ten solutions.

Connection between the integral and fractional solutions, §§ 190–196.

§ 190. Leonardo's problem was purely a question of partitions, and the apples were sold in two different markets, and at two different prices, both of which were integers. In the other form, which would not be much later in date if the manuscript quoted in § 2 is of the thirteenth century, some of the eggs were sold at so many to the penny and those that were left over at an integral number of pence each. The question was subsequently made more puzzling by being expressed in a misleading manner, the words implying that each woman sold her eggs at one price. It would have been more natural for both prices to have been of the same character, *i.e.* so that each woman sold her eggs for an integral number of pence: but if as many as possible were sold at so many for a penny, and some were left over, this afforded a reason for the latter being sold subsequently at

* "Récreations mathématiques et physiques . . . Par M. Ozanam . . . Nouvelle édition, totalement refondue et considérablement augmentée par M. de M*** . . ." (Paris. 1790). In 4 volumes. It is problem xii. and occupies pp. 199–204 of vol. i. Poggendorff gives 1778 as the date of the first edition edited by Montucla.

An English translation by Hutton was published at London in 1803 under the title "Recreations in mathematics and natural philosophy . . . First composed by M Ozanam . . . Lately recomposed, and greatly enlarged . . . by . . . M. Montucla. And now translated into English, and improved . . . by Charles Hutton . . . in four volumes . . ." In this translation the problem occupies pp. 196–201 of vol. i. In the edition of 1840 (which consists of one volume only) it occupies pp. 88–90.

† The question is given in the form "Une femme a vendu 10 perdrix au marché, une seconde en a vendu 25, & une troisieme en a vendu 30, & toutes au même prix. Au sortir du marché elles se questionnent sur l'argent qu'elles en rapportent, & il se trouve que chacune rapporte la même somme. On demande à quel prix & comment elles ont vendu". In Hutton's translation (1803) eggs are substituted for partridges, and the sous become pence.

a higher price in the same market. In the original solution which persisted for so long, the number of pence brought back by each woman was 10, the same as the number of eggs sold by the woman with the fewest eggs: and if the problem had required that this sum or any sum less than 50 pence was to be brought back by each, some of the eggs must have been sold at so many for a penny, or at so many for two pence, &c.

Of the two kinds of solutions, viz. (i) in which some of the eggs are sold at so many for a penny and the rest for an integral number of pence each, and (ii) in which at both sales each egg is sold for an integral number of pence, each solution of the first kind gives rise to a solution of the second kind (though several may give rise to the same solution of the second kind); but only some of the solutions of the second kind give rise to solutions of the first kind.

The fractional solutions are made integral by multiplying the second modulus by the denominator of the first, the fixed-sum being also increased in the same proportion. Thus the solutions

$$\begin{array}{r|rcrc|l}
10 & 7\ ; & \frac{1}{7} : & 3\ ; & 3 & 10, \\
30 & 28\ ; & ,, : & 2\ ; & ,, & ,, \\
50 & 49\ ; & ,, : & 1\ ; & ,, & ,,
\end{array}
\qquad
\begin{array}{r|rcrc|l}
10 & 7\ ; & 1 : & 3\ ; & 21 & 70 \\
30 & 28\ ; & ,, : & 2\ ; & ,, & ,, \\
50 & 49\ ; & ,, : & 1\ ; & ,, & ,,
\end{array}$$

are equivalent; but if the eggs had been sold for one penny and 21 pence each there would have been no reason why the numbers of eggs sold at the first price should have been 7, 28, and 49 rather than any other of the eight possible sets of numbers (§ 68, p. 49).

§ 191. Considering in more detail the two kinds of solution, it is evident that a system of the first kind, in which the representations are of the form (§ 93, p. 68)

$$kr + m \,\Big|\, kr\ ;\quad \frac{1}{r} :\quad m\ ;\quad s \,\Big|\, k + ms \text{............(i)},$$

gives rise to an integral system in which the representations are of the form

$$kr + m \,|\, kr\ ;\quad 1 :\quad m\ ;\quad sr \,|\, r\,(k + ms),$$

and that an integral representation $|\, \alpha_1\ ;\ \lambda :\ \beta_1\ ;\ \mu \,|$ gives rise to a representation of the form (i) only when $\lambda = 1$ and α_1 and μ have a common factor. Thus, selecting from § 68 (p. 49)

all the systems in which (when the common factors are divided out from the two moduli) the first modulus is 1 and α_1 and μ have a common factor; and, writing down only the first line of each system,* we obtain the following eight systems

$10|0;\ 1:\ 10;\ 21|210,\quad 10|3;\ 1:\ 7;\ 21|150,\quad 10|6;\ 1:\ 4;\ 21|90,$

$10|7;\ 1:\ 3;\ 21|\ 70,\quad 10|0;\ 1:\ 10;\ 11|110,\quad 10|0;\ 1:10;\ \ 6|60,$

$10|2;\ 1:\ 8;\ \ 6|\ 50,\quad 10|0;\ 1:\ 10;\ \ 5|\ 50,$

from which we derive the following 12 systems of the form (i)

$10|0;\ \tfrac{1}{3}:10;\ 7|70,\quad 10|0;\ \tfrac{1}{7}:10;\ 3|30,\quad 10|0;\ \tfrac{1}{21}:10;\ 1|10,$

$10|3;\ \tfrac{1}{3}:\ 7;\ 7|50,\quad 10|6;\ \tfrac{1}{3}:\ 4;\ 7|30,\quad 10|7;\ \tfrac{1}{7}:\ 3;\ 3|10,$

$10|0;\ \tfrac{1}{11}:10;\ 1|10,\quad 10|0;\ \tfrac{1}{2}:10;\ 3|30,\quad 10|0;\ \tfrac{1}{3}:10;\ 2|20,$

$10|0;\ \tfrac{1}{6}:10;\ 1|10,\quad 10|2;\ \tfrac{1}{2}:\ 8;\ 3|25,\quad 10|0;\ \tfrac{1}{5}:10;\ 1|10,$

which are the first lines of the three-line systems with a fractional modulus in § 61†.

If the two moduli in the first line of a system are $\frac{1}{r}$ and s, the system may be completed by adding $\frac{rsd}{rs-1}$ in the first column and subtracting $\frac{d}{rs-1}$ in the third, where d is the common difference between the numbers represented.

§ 192. Similarly from the systems for the numbers 10, 15, 20, 25, 30 in § 188 the first lines of those for which α_1 and μ have a common factor are

$10|2;\ 1:\ 8;\ 6|50,\quad 10|3;\ 1:\ 7;\ 6|45,$

$10|4;\ 1:\ 6;\ 6|40,\quad 10|6;\ 1:\ 4;\ 6|30,$

which give the fractional systems whose first lines are

$10|2;\ \tfrac{1}{2}:\ 8;\ 3|25,\quad 10|3;\ \tfrac{1}{3}:\ 7;\ 2|15,\quad 10|4;\ \tfrac{1}{2}:\ 6;\ 3|20,$

$10|6;\ \tfrac{1}{2}:\ 4;\ 3|15,\quad 10|6;\ \tfrac{1}{3}:\ 4;\ 2|10,\quad 10|6;\ \tfrac{1}{6}:\ 4;\ 1|\ 5.$

---

* If in the first line $|\alpha_1;\lambda:\beta_1;\mu|$ of any system α_1 has a common factor with μ, so also will $\alpha_2, \alpha_3, \ldots$, for $\alpha_2=\alpha_1+\mu d'$, $\alpha_3=\alpha_1+2\mu d'$, ... where d' is the quotient of d, the common difference of the number, when divided by $\mu-\lambda$. Similarly if β_1 has a common factor with λ, so also will $\beta_2, \beta_3, \ldots$ for $\beta_2=\beta_1-\lambda d'$, $\beta_3=\beta_1-2\lambda d'$, ... When α_1 is 0, we may regard it as divisible by μ and therefore also by all its factors.

† The difference in notation is to be noticed, *e.g.* $|1, 3:7;7|$ is the same as $|3;\tfrac{1}{3}:7;7|$.

§ 193. So far only fractional systems have been considered in which the first modulus is of the form $\frac{1}{r}$ and the second modulus is an integer s, because these are the only solutions which were considered admissible for the egg question (§ 1). Thus the fractional systems considered have been only those which were derivable from integral systems having as first line $|\alpha_1;\ \lambda:\ \beta_1;\ \mu|$ in which λ was unity and α_1 and μ had a common factor: but we also obtain fractional solutions when λ is not unity (λ and μ being supposed to be prime to each other) if α_1 and μ have a common factor, and also if β_1 and λ have a common factor, and therefore also in which both of these conditions occur.

Thus from the systems in § 68 (p. 50) whose first lines are

$$10\,|\,0\ ;\ 3:\ 10\ ;\ 23\,|\,230,\quad 10\,|\,4\ ;\ 3:\ 6\ ;\ 23\,|\,150,$$

we derive the fractional systems whose first lines are

$$10\,|\,0\ ;\ \tfrac{3}{23}:\ 10\ ;\ 1\,|\,10,\quad 10\,|\,4\ ;\ 1:\ 6\ ;\ \tfrac{23}{3}\,|\,50,$$

and which can be completed by adding 23 in the first column and subtracting 3 in the third; and from the 2-line systems in the same section whose first lines are

$$10\,|\,0\ ;\ 2:\ 10\ ;\ 7\,|\,70,\quad 10\,|\,2\ ;\ 2:\ 8\ ;\ 7\,|\,60,$$

we derive the fractional systems whose first lines are

$$10\,|\,0\ ;\ \tfrac{2}{7}:10\ ;\ 1\,|\,10,\quad 10\,|\,0\ ;\ 1:10\ ;\ \tfrac{7}{2}\,|\,35,\quad 10\,|\,2\ ;\ 1:8\ ;\ \tfrac{7}{2}\,|\,30,$$

and which can be completed by adding 28 in the first column and subtracting 8 in the third.

§ 194. As an example in which both α_1 and μ, and β_1 and λ have a common factor, we may take from the systems in § 181 (in which the numbers represented are 20, 30, 40) those in which the first lines are

$$20\,|\,2;\ 3:\ 18;\ 8\,|\,150,\quad 20\,|\,8;\ 3:\ 12;\ 8\,|\,120,$$

and from these systems we may derive the fractional systems whose first lines are

$$20\,|\,2;\ \tfrac{1}{2}:\ 18;\ \tfrac{4}{3}\,|\,25,\quad 20\,|\,8;\ \tfrac{1}{2}:\ 12;\ \tfrac{4}{3}\,|\,20,$$

$$20\,|\,8;\ \tfrac{1}{4}:\ 12;\ \tfrac{2}{3}\,|\,10,\quad 20\,|\,8;\ \tfrac{1}{8}:\ 12;\ \tfrac{1}{3}\,|\ 5.$$

The systems may be completed by adding 16 in the first column and subtracting 6 in the third.

As an example in which the fractional representation is of the form (§ 93),

$$kr+mt \,\Big|\, kr\,;\ \frac{q}{r}:\ mt\,;\ \frac{s}{t} \,\Big|\, kq+ms \ldots\ldots\ldots \text{(iv)}$$

(none of the letters being unity), we may take the system

$$\begin{array}{r|rrrr|l} 13 & 3; & 4: & 10; & 15 & 162 \\ 24 & 18; & ,,: & 6; & ,, & ,, \\ 35 & 33; & ,,: & 2; & ,, & ,,\,, \end{array}$$

which gives

$$\begin{array}{r|rrrr|l} 13 & 3; & \frac{2}{3}: & 10; & \frac{5}{2} & 27 \\ 24 & 18; & ,,: & 6; & ,, & ,, \\ 35 & 33; & ,,: & 2; & ,, & ,,\,. \end{array}$$

In general, the system whose first line is (iv) may be completed by adding $\dfrac{rsd}{rs-qt}$ in the first column and subtracting $\dfrac{qtd}{rs-qt}$ in the third column, d being the common difference of the numbers represented.

§ 195. It will have been seen that the most complete solution of the partition question is afforded by the integral systems, from which all the fractional systems of the different forms can be directly derived.

§ 196. I have now completed the historical account of the partition problems, all substantially the same but different in form, of which solutions were given by Leonardo, Widman, Tagliente, Tartaglia, Bachet, Ozanam, &c.; and I have added mathematical developments arising directly from these solutions or suggested by them. Besides the developments given in this paper I have worked out various results connected with similar problems, but as they have not so close a relation to the history of the subject, I reserve them for separate papers. There still remain a variety of partition questions dependent upon the same principles, but subject to different restrictions, which seem to deserve attention.

CONTENTS OF THE PAPER.

PART I.

GENERALISATION OF WIDMAN'S QUESTION. SYSTEMS OF NUMBERS WHICH WHEN DIVIDED BY p ARE SUCH THAT THE SUM OF THE QUOTIENT AND μ TIMES THE REMAINDER IS THE SAME, p AND μ BEING GIVEN NUMBERS.

PART II.

PARTITIONS OF NUMBERS INTO THE FORM $p\alpha+\beta$ WHERE $\alpha+\mu\beta$ IS THE SAME.

PART III.

PART IV.

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PART V.

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NOTES ON THE CONTENTS OF THE PAPER.

The strictly historical matter in this paper is contained in the following sections:—

§§ 1–6, 25. Early MSS., Widman, Tagliente, Blasius (14th to 16th centuries).

§§ 73–74. Benedetto, Giovanni del Sodo, Ghaligai (15th and 16th centuries).

§§ 75, 79, 112. Tartaglia (16th century).

§§ 115–143. Leonardo Pisano (13th century).

§§ 173–175, 182–184. Bachet, Van Etten, Ozanam (17th century).

As several distinct partition problems (suggested by the historical matter) are treated mathematically in this paper, it is convenient to specify them here and give references to the sections of the paper where they are to be found.

The problem to which the first 64 sections relate is the partitionment of a series of numbers in arithmetical progression into parts of the form $p\alpha+\beta$ subject to the condition that $\alpha+\mu\beta$ is the same for all, p and μ being given numbers. In §§ 1–36 the problem is limited by the further condition that α is to be the quotient and β the remainder when the number is divided by p; that is to say, the condition is that $\beta<p$: but in §§ 37 47 this restriction is discarded. This problem was suggested by the puzzle-question of Widman, Tagliente, &c., who took as the numbers to be partitioned 10, 30, 50, and 20, 40, 60. Full lists of systems of partitions for these numbers are given in §§ 61 and 63, and a similar list for the numbers 33, 42, 51, having the common difference 9, is given in §§ 40–41.

The problem in which the numbers are partitioned into the form $\alpha+\beta$ where $\lambda\alpha+\mu\beta$ is to be the same, λ and μ being integers, is first considered in §§ 65–69; the case of $p=1$ in the preceding problem corresponding to the case $\lambda=1$ of this problem. The systems of partitions for 10, 30, 50 with integral values of λ and μ are given in § 68.

In § 83 a new problem, suggested by Tartaglia's questions, is considered, viz. the partitionment of a, $3a$, $5a$ into the form $\alpha+\beta$ where $\lambda\alpha+\mu\beta$ is the same for all three numbers and is equal to a, all values of λ and μ, fractional as well as integral, being admissible. This investigation, which occupies §§ 83–92, is continued in §§ 93–111, subject to the additional restriction that $\lambda\alpha$ and $\mu\beta$ are to be integral.

In §§ 144–154 the problem considered is the partitionment of n_1 and n_2 into two parts $\alpha_1+\beta_1$ and $\alpha_2+\beta_2$ subject to the condition that the sums $\lambda\alpha_1+\mu\beta_1$ and $\lambda\alpha_2+\mu\beta_2$ are the same, λ and μ being integers; and in §§ 155–167 the condition is, not that these sums should be equal, but that one should be a multiple of the other. These investigations were suggested by Leonardo's treatment of these questions. Following Leonardo, the values of n_1 and n_2 taken in the examples are 12 and 32.

In §§ 170–172 three-line systems are considered and systems are obtained in which the numbers are 10, 30, 50 and the sums are in the proportion of 1, 2, 3 and 3, 2, 1 and 5, 3, 1.

Special attention has to be paid to the distinction between the notations used for the systems in the earlier and later portions

of the paper. These notations are described in § 12 and also in § 72. In $|\alpha, \lambda : \beta ; \mu|$, where there is a comma between α and λ and a semi-colon between β and μ, the number represented is $\alpha\lambda + \beta$ and the sum is $\alpha + \beta\mu$: but in $|\alpha ; \lambda : \beta ; \mu|$, where there is a semi-colon between α and λ and also between β and μ, the number is $\alpha + \beta$ and the sum $\alpha\lambda + \beta\mu$. Thus in forming the number the comma indicates union by multiplication, and the semi-colon indicates rejection, while they have the reverse meanings in the formation of the sum. The colon in the middle indicates addition in both notations.

In §§ 1-64 the first notation (with the comma between the first two numbers) is alone used, and in the subsequent portion of the paper the second notation is alone used. The difference in the notations in the two portions of the paper is due to the different character of the problems considered. The notation $|\alpha, \lambda : \beta ; \mu|$ is equivalent to $\left|\alpha\lambda ; \frac{1}{\lambda} : \beta ; \mu\right|$ which differs from $|\alpha\lambda ; 1 : \beta ; \lambda\mu|$ only in the sum, the number represented being unaltered.

SUR UNE ESPÈCE DE SÉRIES TRIGONOMÉTRIQUES.

Par *G. Boev* (Saratov, Russie).

Les séries dont le terme général est $a_n \operatorname{cosec} nx$ ou $a_n \cot nx$ ne s'emploient presque pas et sont peu étudiées. Elles se diffèrent des séries trigonométriques ordinaires par la propriété essentielle du terme général d'admettre un pol dépendant de n. Tenant compte de cette propriété, on peut espérer que les séries de la forme indiquée seront utiles pour la construction des fonctions analytiques *périodiques*, qui possèdent une *ligne singulière*. Quelques exemples particuliers confirmeront cette idée. Je vais en citer trois:

$$\Sigma \frac{1}{(2n-1)} \operatorname{cosec}(2n-1)x, \quad \Sigma \frac{(-1)^n}{n} \operatorname{cosec} nx, \quad \Sigma \frac{(-1)^n}{n} \cot nx.$$

Ces séries-ci convergent hors de l'axe réel et représentent des fonctions qui s'expriment par la fonction modulaire $k^2(\tau)$ bien connue.

Considérons d'abord la série

$$\phi = \Sigma \frac{1}{r} \operatorname{cosec} rx \qquad (r = 1, 3, 5, \ldots).$$

Posons $e^{ix} = q$ et faisons les transformations que voici:

$$\phi = -2i\Sigma \frac{1}{r} \frac{q^r}{1-q^{2r}} = -2i\Sigma_r \frac{1}{r} \Sigma_n q^{r(2n-1)}$$

$$= i\Sigma \log \frac{1-q^{2n-1}}{1+q^{2n+1}} = i \log \Pi \frac{1-q^{2n-1}}{1+q^{2n+1}}.$$

Les séries qui y figurent convergent pour $|q|<1$. Le produit infini, que nous venons d'obtenir, représente la fonction modulaire $\sqrt[4]{\{k'(\tau)\}}$ d'Hermite, ou $\tau = \frac{1}{\pi i}.\log q$. Donc nous avons

$$\phi = \frac{i}{4} \log k'\left(\frac{x}{\pi}\right).$$

Prenons maintenant la série

$$\psi = \Sigma \frac{(-1)^n}{n} \operatorname{cosec} nx.$$

$$\psi = -2i\Sigma \frac{(-1)^n}{n} \frac{q^n}{1-q^{2n}} = -2i\Sigma_m\Sigma_n \frac{(-1)^n}{n} q^{(2m-1)n}$$

$$= 2i\Sigma \log (1+q^{2m-1}).$$

Mais on sait que

$$\sqrt[12]{(kk')} = \sqrt[3]{(2)}\, q^{\frac{1}{24}} \Pi \frac{1}{1+q^{2m-1}}.$$

Il en suit:

$$\psi = \frac{i}{3} \log 2 - \frac{x}{12} - \frac{i}{6} \log kk'\left(\frac{x}{\pi}\right).$$

Enfin prenons la série

$$\chi = \Sigma \frac{(-1)^n}{n} \cot nx.$$

$$\chi = i \log 2 + \Sigma \frac{(-1)^n}{n} (\cot nx + i)$$

$$= i \log 2 + \Sigma \frac{(-1)^n}{n} \frac{\cos nx + i \sin nx}{\sin nx}$$

$$= i \log 2 + 2i\Sigma \frac{(-1)^n}{n} \frac{q^{2n}}{1-q^{2n}} = i \log 2 + 2i\Sigma_n \frac{(-1)^n}{n} \Sigma_m q^{2mn}$$

$$= i \log 2 - 2i\Sigma \log(1+q^{2m}) = i \log 2 - 2i \log \Pi\, (1+q^{2m}).$$

En tenant compte des identités connues

$$\sqrt[4]{(k)} = \sqrt{(2)}\, q^{\frac{1}{8}} \Pi \frac{1+q^{2m}}{1+q^{2m-1}}, \qquad \sqrt[12]{(kk')} = \sqrt[6]{(2)}\, q^{\frac{1}{24}} \Pi \frac{1}{1+q^{2m-1}},$$

nous arrivons au résultat suivant

$$\chi = \frac{i}{3}\log 2 + \frac{x}{6} + \frac{i}{3}\log \frac{k(x/\pi)}{\sqrt{\{k'(x/\pi)\}}}.$$

Les transformations des séries que nous venons de faire, sont légitimes à cause de la convergence uniformes des développements de $\log k$, $\log k'$ et de $\log(1 \pm q^{\nu})$ sous la condition d'avoir $|q| < 1$, c'est à dire, que x reste dans le demiplan supérieur et hors de l'axe réel. Ce dernier présente pour les fonctions ϕ, ψ, χ une ligne singulière.

Les exemples considérés nous montrent, que le problème de développement en série des cosécantes d'une fonction analytique arbitraire, mais périodique et à ligne singulière, a raison d'être posé.

ON THE EXPRESSION OF AN INFINITE PRODUCT IN $G(z)$ FUNCTIONS.

By *N. Koshliakov.*

In Whittaker's *Course of Modern Analysis* we find an interesting formula, which expresses in gamma-functions the infinite product

$$P = \prod_{n=1}^{\infty} \left\{ \frac{(n-a_1)(n-a_2)\dots(n-a_k)}{(n-b_1)(n-b_2)\dots(n-b_k)} \right\}$$

subject to the condition

$$a_1 + a_2 + \dots + a_k = b_1 + b_2 + \dots + b_k;$$

namely
$$P = \frac{\Gamma(1-b_1)\,\Gamma(1-b_2)\dots\Gamma(1-b_k)}{\Gamma(1-a_1)\,\Gamma(1-a_2)\dots\Gamma(1-a_k)}.$$

The object of the present note is to deduce an analogous formula for the infinite product

$$Q = \prod_{n=1}^{\infty} \left\{ \frac{(n-a_1)(n-a_2)\dots(n-a_k)}{(n-b_1)(n-b_2)\dots(n-b_k)} \right\}^n \quad \dots\dots\dots(1)$$

subject to the conditions

$$\left.\begin{aligned} a_1 + a_2 + \ldots + a_k &= b_1 + b_2 + \ldots + b_k \\ a_1^2 + a_2^2 + \ldots + a_k^2 &= b_1^2 + b_2^2 + \ldots + b_k^2 \end{aligned}\right\} \quad \ldots\ldots(2).$$

This formula is found to be

$$Q = \frac{G(1-a_1)\,G(1-a_2)\ldots G(1-a_k)}{G(1-b_1)\,G(1-b_2)\ldots G(1-b_k)} \ldots\ldots\ldots(3),$$

where $G(z)$ is a function satisfying the functional equation

$$G(z+1) = \Gamma(z)\,G(z)$$

subject to the condition $G(1)=1$, and investigated by Barnes, Glaisher, and Alexejewsky in a series of memoirs.*

Let us show first that the infinite product (1) is absolutely convergent. In fact, by using the relations (2) we have

$$\left\{\left(1-\frac{a_1}{n}\right)\left(1-\frac{a_2}{n}\right)\ldots\left(1-\frac{a_k}{n}\right)\left(1-\frac{b_1}{n}\right)^{-1}\left(1-\frac{b_2}{n}\right)^{-1}\ldots\left(1-\frac{b_k}{n}\right)^{-1}\right\}^n$$

$$= \left\{1 + O\left(\frac{1}{n^3}\right)\right\}^n = 1 + O\left(\frac{1}{n^2}\right),$$

whence results the required property of our infinite product.

It follows from this absolute convergence and from the conditions (2) that the expression for Q may be presented in the form

$$Q = \frac{\prod\limits_1^\infty \left\{\left(1-\frac{a_1}{n}\right) e^{\frac{a_1}{n}+\frac{1}{2}\frac{a_1^2}{n^2}}\right\}^n \ldots \prod\limits_1^\infty \left\{\left(1-\frac{a_k}{n}\right) e^{\frac{a_k}{n}+\frac{1}{2}\frac{a_k^2}{n^2}}\right\}^n}{\prod\limits_1^\infty \left\{\left(1-\frac{b_1}{n}\right) e^{\frac{b_1}{n}+\frac{1}{2}\frac{b_1^2}{n^2}}\right\}^n \ldots \prod\limits_1^\infty \left\{\left(1-\frac{b_k}{n}\right) e^{\frac{b_k}{n}+\frac{1}{2}\frac{b_k^2}{n^2}}\right\}^n}.$$

Taking into account one of the fundamental formulas of the theory of G-functions

$$G(z+1) = (2\pi)^{z/2} \cdot e^{-\frac{z(z+1)}{2} - 8\frac{z^2}{2}} \prod_1^\infty \left\{\left(1+\frac{z}{n}\right) e^{-\frac{z}{n}+\frac{1}{2}\frac{z^2}{n^2}}\right\}^n,$$

we obtain the required relation (3).

We shall point out, in conclusion, that the fundamental formula of the theory of G-functions

$$\phi(z) - \phi(1) = (z-1)\,\psi(z) - z + 1 \ \ldots\ldots\ldots\ldots\ldots(4),$$

* Barnes, *Quarterly Journal*, vol. xxxi.; Glaisher. *Messenger*, vols. vi., vii., xxiii., xxiv.; Alexejewsky *eipziger Sitzungsber.* 1894.

where $\phi(z) = D_z^{(1)} \log G(z), \quad \psi(z) = D_z^{(1)} \log \Gamma(z),$

which is deduced by some authors in a rather complicated manner, can be obtained by means of very simple reasoning.

From the equalities

$$\Gamma(z+1) = z\Gamma(z), \quad G(z+1) = \Gamma(z)\, G(z),$$

it follows that

$$\log \Gamma(z) = \Sigma \log z, \quad \log G(z) = \Sigma \log \Gamma(z) = \Sigma\Sigma \log z,$$

$$\psi(z) = \Sigma \frac{1}{z}, \qquad \phi(z) = \Sigma\psi(z) = \Sigma\Sigma \frac{1}{z}.$$

By substituting the function $\frac{1}{z}$ in the place of $f(z)$ in the equality

$$\Sigma\Sigma f(z) = (z-1)\, \Sigma f(z-1) - \Sigma\left[(z-1) f(z)\right],$$

which may be easily proved, we come at once to the formula (4).

Simferopol,
January 20th, 1924.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

LVII.

On Fourier transforms.

1. THE theory of 'Fourier transforms' is due primarily to Plancherel.* Plancherel proved that, if $f(x)$ is integrable† over any finite interval $(0, X)$, and its square is integrable over $(0, \infty)$, then a function $g(x)$ exists which has the same properties, and is connected with $f(x)$ by the reciprocal relations

$$(1.11) \quad f(x) = \sqrt{\left(\frac{2}{\pi}\right)}\, \frac{d}{dx} \int_0^\infty \frac{\sin xy}{y}\, g(y)\, dy,$$

$$(1.12) \quad g(x) = \sqrt{\left(\frac{2}{\pi}\right)}\, \frac{d}{dx} \int_0^\infty \frac{\sin xy}{y}\, f(y)\, dy,$$

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* M. Plancherel, 'Contribution à l'étude de la représentation d'une fonction arbitraire par des intégrales définies', *Rend. Circ. Mat. Palermo*, 30 (1910), pp. 289–335.

† In the sense of Lebesgue.

for almost all values of x. Further

$$(1.13) \qquad \int_0^\infty \{f(x)\}^2 dx = \int_0^\infty \{g(x)\}^2 dx.$$

The formulae (1.1) reduce to the well-known formulae of Fourier when differentiation under the sign of integration is permissible, and assert, in a symmetrical form, the reciprocity between two functions implied in 'Fourier's integral theorem'.

Plancherel obtained the formulae (1.1) as corollaries of a very general theory, and his actual deduction of them is somewhat artificial. More recently Titchmarsh * has investigated them in a more direct and natural manner, and has established the corresponding formulae associated with Hankel's generalisation of Fourier's theorem.

In this note I give an alternative investigation, which differs from Titchmarsh's in that all relics of the general theory of 'orthogonal functions' have disappeared.

2. The integral

$$(2.1) \qquad \psi(x) = \int_0^\infty \frac{\sin xy}{y} f(y)\, dy$$

is absolutely convergent, since

$$\left(\int_0^\infty \left| \frac{\sin xy}{y} f(y) \right| dy\right)^2 \leq \int_0^\infty \left(\frac{\sin xy}{y}\right)^2 dy \int_0^\infty \{f(y)\}^2 dy.$$

If

$$(2.2) \qquad \Delta(x, h) = \frac{\psi(x+h) - \psi(x-h)}{2h} = \int_0^\infty \frac{\sin hy}{hy} \cos xy\, f(y)\, dy,$$

we have

$$(2.3) \qquad \Delta(x, h)\, \Delta(x, h')$$
$$= \int_0^\infty \int_0^\infty \frac{\sin hy}{hy} \frac{\sin h'y'}{h'y'} \cos xy \cos xy' f(y) f(y')\, dy\, dy',$$

the double integral being absolutely convergent, and uniformly convergent in x. Hence

$$(2.4) \qquad \int_0^X \Delta(x, h)\, \Delta(x, h')\, (X - x)\, dx$$
$$= \int_0^\infty \int_0^\infty \frac{\sin hy}{hy} \frac{\sin h'y'}{h'y'} f(y) f(y')\, dy\, dy' \int_0^X (X-x) \cos xy \cos xy'\, dx$$
$$= \int_0^\infty \int_0^\infty \frac{\sin hy}{hy} \frac{\sin h'y'}{h'y'} \left\{ \frac{\sin^2 \tfrac{1}{2} X (y + y')}{(y + y')^2} + \frac{\sin^2 \tfrac{1}{2} X (y - y')}{(y - y')^2} \right\}$$
$$\times f(y) f(y')\, dy\, dy'.$$

* E. C. Titchmarsh, 'Hankel transforms', *Proc. Camb. Phil. Soc.*, 21 (1923), pp. 463–473.

3. Lemma A. *There is a constant K such that*

$$(3.1)\qquad \int_0^\infty \int_0^\infty |f(y)|\,|f(y')|\,\frac{\sin^2 m\,(y \pm y')}{(y \pm y')^2}\,dy\,dy' < Km$$

for all real values of m.

We define $f(y)$ for negative y so that it shall be even. Then

$$\int_{-\infty}^\infty \int_{-\infty}^\infty |f(y)|\,|f(y')|\,\frac{\sin^2 m\,(y \pm y')}{(y \pm y')^2}\,dy\,dy'$$

$$= \int_{-\infty}^\infty |f(y)|\,dy \int_{-\infty}^\infty |f(y+u)|\left(\frac{\sin mu}{u}\right)^2 du$$

$$= \int_{-\infty}^\infty \left(\frac{\sin mu}{u}\right)^2 du \int_{-\infty}^\infty |f(y)|\,|f(y+u)|\,dy$$

$$\leqq \int_{-\infty}^\infty \left(\frac{\sin mu}{u}\right)^2 du \sqrt{\left\{\int_{-\infty}^\infty \{f(y)\}^2 dy \int_{-\infty}^\infty \{f(y+u)\}^2 dy\right\}}$$

$$= \pi m \int_{-\infty}^\infty \{f(y)\}^2\,dy < Km.$$

4. It follows that the integral on the right of (2.4) is for any fixed value of X, uniformly convergent in a rectangle

$$-h_0 \leqq h \leqq h_0, \quad -h_0' \leqq h' \leqq h_0';$$

and therefore that

$$(4.1)\qquad \lim_{h,\,h' \to 0} \int_0^X \Delta(x, h)\,\Delta(x, h')\,(X - x)\,dx$$

$$= \int_0^\infty \int_0^\infty \left\{\frac{\sin^2 \tfrac{1}{2}X(y + y')}{(y + y')^2} + \frac{\sin^2 \tfrac{1}{2}X(y - y')}{(y - y')^2}\right\} f(y)\,f(y')\,dy\,dy'.$$

If we write, for shortness, $\Delta(x, h) = \Delta$, $\Delta(x, h') = \Delta'$, the three integrals

$$\int_0^X \Delta^2 (X - x)\,dx, \quad \int_0^X \Delta\Delta' (X - x)\,dx, \quad \int_0^X \Delta'^2 (X - x)\,dx$$

tend to the same limit when h and h' tend to zero, and therefore

$$(4.2)\qquad \int_0^X (\Delta - \Delta')^2 (X - x) = \int_0^X (\Delta^2 - 2\Delta\Delta' + \Delta'^2)(X - x)\,dx \to 0.$$

It follows that $\Delta(x, h)\sqrt{(X - x)}$ 'converges in mean'* to a function $\chi(x)\sqrt{(X - x)}$ integrable, with its square, in $(0, X)$;

* See Plancherel, *l.c.*, p. 4. The idea of 'convergence en moyenne', which is fundamental in this theory, is due to E. Fischer ('Sur la convergence en moyenne', *Comptes Rendus*, 13 May, 1907).

and that

$$\int_0^X (\Delta - \chi)^2 (X - x)\, dx \to 0 \tag{4.3}$$

and

$$\int_0^X \Delta^2 (X - x)\, dx \to \int_0^X \chi^2 (X - x)\, dx, \tag{4.4}$$

when $h \to 0$.

If $0 < \xi < X$,

$$\int_0^\xi (\Delta - \chi)^2 dx \leqq \frac{1}{X - \xi} \int_0^\xi (\Delta - \chi)^2 (X - x)\, dx \to 0. \tag{4.5}$$

Hence Δ converges in mean to χ in $(0, \xi)$, that is to say in any finite interval, and χ and χ^2 are integrable in any finite interval. Also

$$\left\{\int_0^\xi (\Delta - \chi)\, dx\right\}^2 \leqq \xi \int_0^\xi (\Delta - \chi)^2\, dx \to 0,$$

$$\lim_{h \to 0} \int_0^\xi \Delta(x, h)\, dx = \int_0^\xi \chi(x)\, dx. \tag{4.6}$$

Next, integrating (2.2) over $(0, \xi)$, we obtain

$$\int_0^\xi \Delta(x, h)\, dx = \int_0^\infty \frac{\sin hy}{hy} \frac{\sin \xi y}{y} f(y)\, dy,$$

and the last integral is uniformly convergent in h. Hence

$$\lim_{h \to 0} \int_0^\xi \Delta(x, h)\, dx = \int_0^\infty \frac{\sin \xi y}{y} f(y)\, dy$$

or

$$\int_0^\xi \chi(x)\, dx = \int_0^\infty \frac{\sin \xi y}{y} f(y)\, dy; \tag{4.7}$$

and

$$\chi(\xi) = \frac{d}{d\xi} \int_0^\infty \frac{\sin \xi y}{y} f(y)\, dy$$

for almost all ξ. If

$$\chi(\xi) = \sqrt{\left(\frac{\pi}{2}\right)}\, g(\xi), \tag{4.8}$$

this is (1.12).

5. We prove next that g^2 is integrable over $(0, \infty)$, and that

$$\int_0^\infty \{g(x)\}^2 dx = \int_0^\infty \{f(x)\}^2 dx. \tag{5.1}$$

We have, from (4.1),

$$\int_0^X \chi^2 (X-x)\,dx \tag{5.2}$$

$$= \int_0^\infty \int_0^\infty \left\{ \frac{\sin^2 \frac{1}{2}X(y+y')}{(y+y')^2} + \frac{\sin^2 \frac{1}{2}X(y-y')}{(y-y')^2} \right\} f(y) f(y')\, dy\, dy'$$

or

$$\frac{1}{X}\int_0^X \chi^2 (X-x)\,dx = \int_0^\infty \{F_1(y, X) + F_2(y, X)\} f(y)\, dy, \tag{5.3}$$

where

$$\begin{cases} F_1(y, X) = \dfrac{1}{X}\displaystyle\int_0^\infty f(y') \dfrac{\sin^2 \frac{1}{2}X(y+y')}{(y+y')^2}\, dy', \\ F_2(y, X) = \dfrac{1}{X}\displaystyle\int_0^\infty f(y') \dfrac{\sin^2 \frac{1}{2}X(y-y')}{(y-y')^2}\, dy'. \end{cases} \tag{5.4}$$

When $X \to \infty$,

$$F_1(y, X) \to 0, \quad F_2(y, X) \to \tfrac{1}{2}\pi f(y), \tag{5.5}$$

for almost all y, by Lebesgue's generalisation of Fejer's theorem. If then we may proceed to the limit under the integral sign, on the right-hand side of (5.3), we obtain

$$\lim \frac{1}{X}\int_0^X \chi^2 (X-x)\,dx = \tfrac{1}{2}\pi \int_0^\infty f^2 dy, \tag{5.6}$$

or, since χ^2 is positive,

$$\int_0^\infty \chi^2 dx = \tfrac{1}{2}\pi \int_0^\infty f^2 dy, \tag{5.7}$$

which is (5.1). It is only necessary, then, to justify the limiting process applied to (5.3). This process will certainly be legitimate if

$$\int_0^\infty \{F_1(y, X)\}^2 dy, \quad \int_0^\infty \{F_2(y, X)\}^2 dy \tag{5.8}$$

are bounded functions of X*.

6. It is plainly sufficient to prove that, if we define $f(y)$ for negative y as in § 3, and write

$$F(y, X) = \frac{1}{X}\int_{-\infty}^\infty f(y') \frac{\sin^2 \frac{1}{2}X(y-y')}{(y-y')^2}\, dy', \tag{6.1}$$

* See, *e.g.*, W. H. Young, 'The application of expansions to definite integrals', *Proc. Lond. Math. Soc.* (2), 9 (1910), pp. 463–485 (p. 469).

then

$$(6.2) \qquad \int_{-\infty}^{\infty} \{F(y, X)\}^2 dy$$

is bounded. But this integral is

$$(6.3) \quad \frac{1}{X^2}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y')f(y'') \frac{\sin^2 \frac{1}{2}X(y-y')}{(y-y')^2} \times \frac{\sin^2 \frac{1}{2}X(y-y'')}{(y-y'')^2} dy'\,dy''$$

$$= \frac{\pi}{2X^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y')f(y'') \frac{X(y'-y'')-\sin X(y'-y'')}{(y'-y'')^3} dy'\,dy'',$$

since

$$(6.4) \quad \int_{-\infty}^{\infty} \frac{\sin^2 m(x-a)}{(x-a)^2}\,\frac{\sin^2 m(x-b)}{(x-b)^2}\,dx = \frac{\pi}{2}\,\frac{m(a-b)-\sin m(a-b)}{(a-b)^3}.$$

But the integral (6.3) is

$$\frac{\pi}{2X}\int_0^X \phi(\xi)\,d\xi,$$

where

$$\phi(\xi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y')f(y'')\frac{1-\cos X(y'-y'')}{(y'-y'')^2} dy'\,dy'',$$

which is bounded, by Lemma A. Hence (6.3) is bounded, which completes the proof of (5.1).

7. We have now established the existence and integrability of g and g^2 and the equations (1.12) and (1.13). It remains to prove (1.11).

We have

$$(7.1) \quad \int_0^{\infty} \Delta(x, h)\frac{\sin xu}{x}\,dx$$

$$= \int_0^{\infty}\frac{\sin xu}{x}\,dx \int_0^{\infty}\frac{\sin hy}{hy}\cos xy\, f(y)\,dy$$

$$= \int_0^{\infty}\frac{\sin hy}{hy} f(y)\,dy \int_0^{\infty}\frac{\sin xu}{x}\cos xy\,dx$$

$$= \tfrac{1}{2}\pi\int_0^{u}\frac{\sin hy}{hy} f(y)\,dy,$$

if the integrations may be inverted. Now

$$\int_0^\infty \frac{\sin hy}{hy} \cos xy f(y)\, dy$$

is uniformly convergent over any finite interval of values of x, so that

$$(7.2) \qquad \int_\alpha^\beta \Delta \frac{\sin xu}{x}\, dx = \int_0^\infty \frac{\sin hy}{hy} f(y)\, dy \int_\alpha^\beta \frac{\sin xu}{x} \cos xy\, dx$$

for $0 \leqq \alpha < \beta$. As the inner integral on the right-hand side is bounded for all α, β, and y, we may write 0 for α and ∞ for β, which completes the proof of (7.1).

Making $h \to 0$ in (7.1), we obtain

$$\int_0^\infty \chi(x) \frac{\sin xu}{x}\, dx = \tfrac{1}{2}\pi \int_0^u f(y)\, dy,$$

and so
$$f(u) = \frac{2}{\pi} \frac{d}{du} \int_0^\infty \chi(x) \frac{\sin xu}{x}\, dx,$$

for almost all u, which is (1.11). This process is legitimate if only

$$(7.3) \qquad \lim_{h \to 0} \int_0^\infty \Delta \frac{\sin xu}{x}\, dx = \int_0^\infty \chi \frac{\sin xu}{x}\, dx.$$

Now
$$\lim_{h \to 0} \int_0^\xi \Delta \frac{\sin xu}{x}\, dx = \int_0^\xi \chi \frac{\sin xu}{x}\, dx$$

for every finite ξ, since Δ converges in mean to χ. It is therefore sufficient, in order to prove (7.3), to show that

$$(7.4) \qquad \int_0^\infty \Delta \frac{\sin xu}{x}\, dx$$

converges uniformly in h.

Suppose that α is large. Then, by (7.2),

$$\left| \int_\alpha^\beta \Delta \frac{\sin xu}{x}\, dx \right| \leqq \left(\int_0^{2u} + \int_{2u}^\infty \right) |f(y)|\, |\phi(\alpha, \beta, y)|\, dy = J_1 + J_2,$$

say, where

$$\phi(\alpha, \beta, y) = \int_\alpha^\beta \frac{\sin xu}{x} \cos xy\, dx.$$

In J_2 we have

$$|\phi(\alpha, \beta, y)| = \left| \tfrac{1}{2} \int_{\alpha(y+u)}^{\beta(y+u)} \frac{\sin z}{z}\, dz - \tfrac{1}{2} \int_{\alpha(y-u)}^{\beta(y-u)} \frac{\sin z}{z}\, dz \right| < \frac{K}{\alpha y},$$

where K is a constant. Hence

$$J_2 < \frac{K}{\alpha} \int_{2u}^{\infty} \frac{|f(y)|}{y} \, dy \to 0,$$

when $\alpha \to \infty$. Also $\phi(\alpha, \beta, y)$ is bounded, and tends to zero, when $\alpha \to \infty$, for every y in $(0, 2u)$ save $y = u$. Hence $J_1 \to 0$. As J_1 and J_2 are independent of h, our conclusion follows.

8. The functions $f(x)$ and $g(x)$ are Fourier *cosine* transforms of one another. There is a similar theory of Fourier *sine* transforms, in which

$$(8.1) \qquad f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int_0^{\infty} \frac{1 - \cos xy}{y} g(y) \, dy,$$

$$(8.2) \qquad g(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int_0^{\infty} \frac{1 - \cos xy}{y} f(y) \, dy,$$

and which may be developed in exactly the same manner.

ON AN INTEGRAL CONNECTED WITH THE THEORY OF PROBABILITY.

By *Prof. W. Burnside.*

DENOTE by x_{ij} $(i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n;\ m > n)$ n sets of m real variables. It is proposed to determine the integral of the differential

$$\prod_{ij} dx_{ij}$$

over the range given by the inequalities

$$a_{jk} \leq \prod_{i=1}^{m} x_{ij} x_{ik} \leq a_{jk} + \delta a_{jk} \qquad (j, k = 1, 2, \ldots, n).$$

In order that there may be real values of the variables satisfying these inequalities, the a's must obviously satisfy certain conditions. In particular, if $i, j, \ldots, k$ is any set of distinct integers chosen from $1, 2, \ldots, n$, then

$$\begin{vmatrix} a_{ii}, & a_{ij}, & \ldots, & a_{ik} \\ a_{ij}, & a_{jj}, & \ldots, & a_{jk} \\ \cdots & \cdots & \cdots & \cdots \\ a_{ik}, & a_{jk}, & \ldots, & a_{kk} \end{vmatrix} > 0.$$

It will be supposed that these conditions are in fact satisfied by the a's.

To establish an induction it is assumed that, when ν is less than n, the value of the integral is

$$k_{m,\nu}\begin{vmatrix} a_{11}, & a_{12}, & \dots, & a_{1\nu} \\ a_{12}, & a_{22}, & \dots, & a_{2\nu} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{1\nu}, & a_{2\nu}, & \dots, & a_{\nu\nu} \end{vmatrix}^{\frac{1}{2}(m-\nu-1)} \delta a_{11}\delta a_{12}\dots\delta a_{\nu-1,\nu}\delta a_{\nu\nu},$$

where $k_{m,\nu}$ is a number depending only on m and ν.

The variables x_{i1} $(i=1, 2, \dots, m)$ have constant values while the integration with respect to the remaining $m(n-1)$ is carried out. Suppose an orthogonal transformation effected on these m variables, such that the set of constant values become

$$x_{11}', \;\; 0, \;\; 0, \;\dots,\; 0.$$

Then x_{11}' is sensibly $\sqrt{(a_{11})}$, and the inequalities affecting the other variables become

$$a_{1i} \leq \sqrt{(a_{11})}\, x_{1i} \leq a_{1i} + \delta a_{1i} \quad (i=2, 3, \dots, n),$$

$$a_{jk} \leq \sum_{i=1}^{m} x_{ij}x_{ik} \leq a_{jk} + \delta a_{jk} \quad (j, k=2, 3, \dots, n).$$

The former of these give

$$dx_{12}dx_{13}\dots dx_{1n} = \frac{\delta a_{12}\delta a_{13}\dots\delta a_{1n}}{a_{11}^{\frac{1}{2}(n-1)}},$$

while the last give

$$a_{jk} - \frac{a_{1j}a_{1k}}{a_{11}} \leq \sum_{i=2}^{m} x_{ij}x_{ik} \leq a_{jk} - \frac{a_{1j}a_{1k}}{a_{11}} + \delta a_{jk} \quad (j, k=2, 3, \dots, n).$$

Hence, in virtue of the assumption that has been made, the integral with respect to the $(n-1)m$ symbols, omitting x_{i1} $(i=1, 2, \dots, m)$, is

$$\prod_{i=1}^{m} dx_{i1} \cdot \frac{\delta a_{12}\delta a_{13}\dots\delta a_{1n}}{a_{11}^{\frac{1}{2}(n-1)}} \times$$

$$k_{m-1,n-1}\begin{vmatrix} a_{22} - \dfrac{a_{12}^2}{a_{11}}, & a_{23} - \dfrac{a_{12}a_{13}}{a_{11}}, & \dots, & a_{2n} - \dfrac{a_{12}a_{1n}}{a_{11}} \\ a_{23} - \dfrac{a_{12}a_{13}}{a_{11}}, & a_{23} - \dfrac{a_{13}^2}{a_{11}}, & \dots, & a_{3n} - \dfrac{a_{13}a_{1n}}{a_{11}} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{2n} - \dfrac{a_{12}a_{1n}}{a_{11}}, & a_{3n} - \dfrac{a_{13}a_{1n}}{a_{11}}, & \dots, & a_{nn} - \dfrac{a_{1n}^2}{a_{11}} \end{vmatrix}^{\frac{1}{2}\{m-1-(n-1)-1\}} \delta a_{22}\delta a_{23}\dots\delta a_{nn}.$$

The determinant in this formula is well known to be equal to

$$\frac{1}{a_{11}}\begin{vmatrix} a_{11}, & a_{12}, & \dots, & a_{1n} \\ a_{21}, & a_{22}, & \dots, & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{1n}, & a_{2n}, & \dots, & a_{nn} \end{vmatrix}.$$

Moreover, the integral of

$$\prod_{i=1}^{m} dx_{i1}$$

over the range given by

$$a_{11} \leq x_{11}^2 + x_{21}^2 + \dots + x_{m1}^2 \leq a_{11} + \delta a_{11}$$

is

$$f(m)\, a_{11}^{\frac{1}{2}(m-2)}\, \delta a_{11},$$

where $f(m)$ is a number depending on m only.

Finally then, on the assumption made when ν is less than n, the value of the integral is

$$k_{m-1,n-1} f(m) \begin{vmatrix} a_{11}, & a_{12}, & \dots, & a_{1n} \\ a_{12}, & a_{22}, & \dots, & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{1n}, & a_{2n}, & \dots, & a_{nn} \end{vmatrix}^{\frac{1}{2}(m-n-1)} \delta a_{11} \delta a_{12} \dots \delta a_{nn}.$$

Hence, if the assumed form is true for $n-1$, it is true for n; and the numerical coefficient is determined by

$$k_{m,n} = f(m)\, k_{m-1,n-1}.$$

This gives

$$k_{m,n} = f(m) f(m-1) \dots f(m-n+2)\, k_{m-n+1,1},$$

and, from the assumption that has been made, $k_{m-n+1,1}$ is $f(m-n+1)$. Moreover, it is well known that

$$f(m) = \tfrac{1}{2}m \frac{\pi^{\frac{1}{2}m}}{\Gamma(1+\frac{1}{2}m)};$$

and therefore

$$k_{m,n} = \frac{\pi^{\frac{1}{4}n(2m-n+1)}\,\Gamma(1+m)}{2^n \Gamma(1+m-n)\, \Gamma\{1+\frac{1}{2}(m-n+1)\}\, \Gamma\{1+\frac{1}{2}(m-n+2)\} \dots \Gamma(1+\frac{1}{2}m)}.$$

THE FORM OF AN ISOLATED ELECTRIC PARTICLE.

By *Dr. H. Bateman.*

§ 1. When a particle of electricity has travelled so far from all other particles that it may be regarded as isolated and free from external forces it presumably tends to assume one of a number of possible limiting types of structure determined by a balance of internal forces. The existence of a definite limiting form when the initial conditions are arbitrary is, of course, hypothetical and the particular limiting form which is ultimately assumed presumably depends on the initial conditions, for experimental evidence points to the conclusion that there are at least two distinct types of electric particle, but seems to exclude the possibility of a change from one type to another. This means that if an electric particle is approaching a limiting form and is then temporarily disturbed by a transient electric field it will, when again left to itself, approach the same limiting form as before, the form being described in each case with the aid of an appropriate set of axes.

In a limiting state a particle of electricity can, presumably, by a suitable choice of space and time co-ordinates, be regarded as travelling uniformly along a rectilinear path without change of form. To a first approximation it may be possible to neglect gravitation and adopt the standpoint of the restricted theory of relativity. Then, when the velocity of the particle is not $\geq c$, where c is the velocity of light, it is possible by means of a relativity transformation to find a reference system in which the particle is at rest and it is in such a system that its form should be described.

In a former paper* it was shown that a balance of internal forces could be obtained in the case of a particle of spherical form whose density of electricity, ρ, is a function only of the distance r from the centre of the spherical boundary. The principles of the conservation of energy and momentum and the conditions of continuity at the boundary were satisfied by adopting a stress-energy-tensor T made up of four parts

$$T = T_e + T_s + T_c + T_m.$$

Within the particle the electrostatic potential ψ was found to be connected with ρ by a relation of type $\psi = b\rho^2$, with a constant factor b, and it was computed that the total energy

* *Mess. of Math.*, vol. lii. (Dec. 1922), p. 116.

could be found accurately to seven places of decimals from the tensor T_m alone.

It has now been proved that the total energy can be derived *exactly* from the tensor T'_m and the proof applies to the two types of electric particle whose existence was indicated by the previous analysis, while the numerical calculation was made for only one. The analysis may be made more general by considering the tensor

$$T = T_e + \kappa\,(T'_s + T'_c + T'_m),$$

where κ is a positive constant. The corresponding relation between ψ and ρ is then $\psi = b\rho^{2\kappa}$.

If $\psi_0\rho_0$ denotes the value of $\psi\rho$ at the boundary $r = a$, the total energy associated with the tensor $T'_e + \kappa\,(T'_s + T'_c)$ is

$$U = (\kappa + \tfrac{1}{2}) \int_0^\infty \left(\frac{d\psi}{dr}\right)^2 4\pi r^2\, dr - 2\kappa\psi_0\rho_0 \cdot \tfrac{4}{3}\pi a^3$$

$$= 4\pi \int_0^\infty f(r)\, dr,$$

where $$f(r) = (\kappa + \tfrac{1}{2})\, r^2 \left(\frac{d\psi}{dr}\right)^2 - \tfrac{2}{3}\kappa \frac{d}{dr}(\psi\rho r^3)$$

$$= (\kappa + \tfrac{1}{2})\, r^2 \left(\frac{d\psi}{dr}\right)^2 - \tfrac{2}{3}\kappa\rho \frac{d}{dr}(\psi r^3) - \tfrac{2}{3}\kappa\psi r^3 \frac{d\rho}{dr}.$$

Now $$\psi = b\rho^{2\kappa}, \qquad \frac{d\rho}{dr} = \frac{\rho}{2\kappa\psi}\frac{d\psi}{dr},$$

and $$\rho = -\frac{d^2\psi}{dr^2} - \frac{2}{r}\frac{d\psi}{dr}.$$

Hence, using primes to denote differentiations with respect to r,

$$f(r) = (\kappa + \tfrac{1}{2})\, r^2\psi'^2 + \tfrac{1}{3}\left(\psi'' + \frac{2}{r}\psi'\right)\left[(2\kappa + 1)\, r^3\psi' + 6\kappa r^2\psi\right]$$

$$= \frac{d}{dr}\left[\frac{2\kappa + 1}{6} r^3\psi'^2 + 2\kappa r^2\psi\psi'\right] + \tfrac{2}{3}(1 - \kappa)\, r^2\psi'^2.$$

The quantity within square brackets is zero when $r = 0$ and $r = \infty$ and is continuous at the boundary. Consequently

$$U = \frac{8\pi}{3}(1 - \kappa) \int_0^\infty r^2\psi'^2\, dr,$$

and is zero only when $\kappa = 1$. The energy arising from the tensor T_m is, on the other hand, $2e\kappa\psi_0$.

To calculate the momentum of the particle for uniform motion with velocity v we must first calculate the integral

$$\int (W - X_x)\, dx\, dy\, dz$$

for the whole of space (in the electrostatic case), using the tensor $T_e + \kappa (T_s + T_c)$.

Now

$$W - X_x = \left(1 - \frac{x^2}{r^2}\right) \psi'^2 + 2\kappa \left[\frac{1}{r} \psi\psi' + \frac{x^2}{r^2} \psi\psi'' - \frac{x^2}{r^3} \psi\psi'\right].$$

Therefore

$$\begin{aligned}
&\int (W - X_x)\, dx\, dy\, dz \\
&= 4\pi \int_0^\infty r^2 dr \left[(1 - \tfrac{1}{3}) \psi'^2 + 2\kappa \left(\frac{1}{r} \psi\psi' + \tfrac{1}{3}\psi\psi'' - \frac{1}{3r} \psi\psi'\right)\right] \\
&= \frac{8\pi}{3} (1 - \kappa) \int_0^\infty r^2 \psi'^2 dr + \frac{8\pi\kappa}{3} \int_0^\infty \frac{d}{dr} (r^2 \psi\psi')\, dr.
\end{aligned}$$

The last integral vanishes and so for the tensor $T_e + \kappa (T_s + T_c)$

$$\int (W - X_x)\, dx\, dy\, dz = U.$$

Now a relativity transformation shows that when the particle is moving with velocity v in the direction of the axis of x its momentum derived from the tensor $T_e + \kappa (T_s + T_c)$ is

$$\frac{v}{c^2 \sqrt{\{1 - (v^2/c^2)\}}} \int (W - X_x)\, dx\, dy\, dz = \frac{Uv}{c^2 \sqrt{\{1 - (v^2/c^2)\}}}.$$

Hence the mass associated with this tensor is $\dfrac{U}{c^2}$ in the stationary state; *i.e.* the mass is proportional to the total energy. This law holds also for the tensor κT_m and so it holds for the complete tensor T.

The case $\kappa = 1$ is of chief interest because $U = 0$ and the stationary mass can be derived simply from the tensor T_m. In this case if we use the tensor $\frac{1}{3} T$ instead of T the stationary mass is given by the usual formula of Lorentz, and there is some prospect of the force exerted by an external field on the particle having its usual value.

So long as the particle is isolated and its velocity is uniform there is no difficulty in this theory. The energy density is everywhere positive and has the usual value outside the electric particle. It is only when we consider external fields and accelerated motion that the difficulties begin and that negative energy appears.

The presence of negative energy in a variable field may indicate that the field has not been correctly represented just

as in hydrodynamics the existence of negative pressure in the irrotational flow of fluid past an obstacle indicates that the mathematical solution does not quite give a correct physical picture of the true flow.

Now in the hydrodynamical case the irrotational flow gives one an idea of the true flow that is very nearly correct except for the region of eddy motion behind the obstacle, and in some cases, at least, the eddy motion can be regarded as a small correction superposed on the irrotational motion.

In the electrical problem we may similarly expect that the solution obtained by calculating the fields by classical methods and introducing negative energy so that the principles of conservation will be fulfilled will be very nearly right, in some regions of space at least. The varying reactions arising from the radiation of positive and negative energy from a particle will generally be very small compared with the static forces, and if they do no work in a complete period, as the analysis sometimes seems to indicate, the orbit of an electric particle in a specified field when determined by the present theory should not differ very much from the actual orbit.

The theory needs correction also to include gravitation and, though it is not likely, this correction may eliminate the negative radiant energy. More probably a larger correction is necessary which may be connected with a failure of the Principle of Superposition when two very strong fields are superposed. There is possibly some electromagnetic entity analogous to an eddy and this leads us to enquire whether all the possible types of electricity have been revealed by our present analysis.

§ 2. It was assumed at the outset that the velocity of the electrified particle was not equal to c. Let us now see if it is possible to satisfy the laws of conservation of energy and momentum when electricity travels with this velocity.

With our tensor T, whether κ is unity or not, the equations of motion (laws of conservation) are satisfied in the case of uniform motion with velocity v if *

$$\left.\begin{array}{r} \rho\left[E_x+\frac{1}{c}(v_y H_z-v_z H_y)\right]+2\kappa\psi\frac{\partial}{\partial x}\left[\rho\sqrt{\left(1-\frac{v^2}{c^2}\right)}\right]=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \rho\,(v\,.\,E)-2\kappa\psi\frac{\partial}{\partial t}\left[\rho\sqrt{\left(1-\frac{v^2}{c^2}\right)}\right]=0 \end{array}\right\}\ldots(1).$$

* In these equations ψ is no longer the electrostatic potential, but a retarded potential for a distribution of density $\rho\sqrt{\left(1-\frac{v^2}{c^2}\right)}$. It reduces to the electrostatic potential when $v=0$.

When $v^2=c^2$ the second terms vanish because both ψ and $\sqrt{\left(1-\frac{v^2}{c^2}\right)}$ are zero. Consequently the equations are satisfied with $\rho \neq 0$ when

$$E+\frac{1}{c}(v \times H)=0 \text{ and } (v.E)=0 \ldots\ldots\ldots.(2),$$

ρ and v being determined by the usual equations

$$\operatorname{curl} H=\frac{1}{c}\left(\frac{\partial E}{\partial t}+\rho v\right), \quad \operatorname{div} E=\rho \ldots\ldots\ldots.(3).$$

If in addition we have the relations

$$H-\frac{1}{c}(v \times E)=0, \quad (v.H)=0 \ldots\ldots\ldots\ldots(4),$$

which are consistent with (2) but not a consequence of them, the field vectors satisfy a set of equations which occurs in Sir Joseph Thomson's theory of Faraday tubes, it being supposed of course that the usual equations

$$\operatorname{curl} E=-\frac{1}{c} \frac{\partial H}{\partial t}, \quad \operatorname{div} H=0$$

are included in our scheme.

To solve these equations we write $M=H+iE$, then we have the relations

$$\operatorname{curl} M=-\frac{i}{c} \frac{\partial M}{\partial t}+\frac{\rho}{c} v,$$

$$\operatorname{div} M=i\rho,$$

$$M+\frac{i}{c}(v \times M)=0.$$

But it has been shown elsewhere that if two vectors M and N are connected by relations of type*

$$c \operatorname{curl} M=\rho v+\frac{\partial N}{\partial t}, \quad \operatorname{div} N=\rho,$$

$$cM=v \times N,$$

then functions γ, α, and β can be found such that

$$N_x=\gamma \frac{\partial(\alpha, \beta)}{\partial(y, z)}, \quad M_x=-\frac{\gamma}{c} \frac{\partial(\alpha, \beta)}{\partial(x, t)}$$

....

* *Proc. London Math. Soc.* (2), vol. xviii. (1919), p. 95.

Hence in our case functions γ, α, and β can be found such that

$$-iM_x = \gamma \frac{\partial(\alpha, \beta)}{\partial(y, z)}, \quad M_x = -\frac{\gamma}{c} \frac{\partial(\alpha, \beta)}{\partial(x, t)}$$

....

and so α and β must be connected by the relations

$$\frac{\partial(\alpha, \beta)}{\partial(y, z)} = \frac{i}{c} \frac{\partial(\alpha, \beta)}{\partial(x, t)} = m_x, \text{ say,}$$

....

which imply that the field specified by the complex vector $m = e - ih$ is a simple radiant field.*

Since $M = \gamma m$ the character of the field specified by the vector M is easily determined and it is found that electricity moves with the velocity of light along the rays of the simple radiant field.

When the relations (4) are not satisfied the field is of a more general character, and general expressions for it have not yet been found.

It should be pointed out that if $\rho \sqrt{\left(1 - \frac{v^2}{c^2}\right)}$ remains constant during the motion of an electric particle, as the relation

$$\frac{d}{dt}\left\{\rho \sqrt{\left(1 - \frac{v^2}{c^2}\right)}\right\} = 0$$

of our last paper seems to indicate, there is no prospect of electricity, that once has the velocity c, changing into a type that can move with a smaller velocity. Indeed, if

$$\rho \sqrt{\left(1 - \frac{v^2}{c^2}\right)} = 0$$

initially because $v^2 = c^2$, it can only be zero when $v^2 \neq c^2$ if $\rho = 0$, *i.e.* if the electric particle has been transformed into "æther". Conversely this argument seems to indicate that electric particles travelling with velocity c can arise only from the æther and not from electric particles such as electrons and protons which move with velocities less than c. The total electric charge associated with a particle moving with velocity c may be zero.

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* Expressions for α and β are easily found and the properties of the field determined. Cf. *Proc. London Math. Soc.* (2), vol. xviii. (1919), p. 95; *Bulletin of the National Research Council*, vol. iv. part 6 (Washington, 1922), pp. 108–112.

When an electric particle moves along a curve with the velocity of light the equations (1) may no longer hold, for they were based on the hypothesis of uniform rectilinear motion. It should be pointed out in this connection that the tensor T_m with components

$$W = \frac{2\psi_0 \rho}{\sqrt{\{1-(v^2/c^2)\}}}, \qquad S_x = \frac{2\psi_0 \rho v_x}{\sqrt{\{1-(v^2/c^2)\}}},$$

$$X_x = -\frac{2\psi_0 \rho v_x^{\,2}}{c^2\sqrt{\{1-(v^2/c^2)\}}}, \quad X_y = -\frac{2\psi_0 \rho v_x v_y}{c^2\sqrt{\{1-(v^2/c^2)\}}}$$

becomes indeterminate if ψ_0 and $\sqrt{\{1-(v^2/c^2)\}}$ become simultaneously zero, but if the ratio of these remains finite there may be finite mass associated with the electric particle and also a finite value of ρ. The stationary mass of the particle will, however, be zero.

This result seems to suggest that if an electric particle of the present type can be formed in the æther, it originates at a place where $\psi = 0$.

In the field of an isolated charge such as an electron or proton there are no points where $\psi = 0$, but when the fields of a positive and negative charge are superposed there certainly are places where $\psi = 0$.

Let us consider a point charge Q describing a circular orbit with uniform velocity v round a stationary point charge O. Let r and R be the radii from Q and O to an arbitrary point P, θ the angle which the direction of r makes with the direction of motion of Q. Then $\psi = 0$ if

$$0 = \frac{1}{R} - \frac{\sqrt{(c^2 - v^2)}}{r(c - v\cos\theta)},$$

while $r = c(t-\tau)$, where t is the time at which a disturbance issuing from Q at time τ becomes effective at P.

When θ is constant the above equation generally represents a sphere. As θ varies from π to 0, the sphere first surrounds Q and becomes larger and larger until when

$$c - v\cos\theta = \sqrt{(c^2 - v^2)}$$

it is a plane. The points at infinity on the cone for which θ has this value are possible points at infinity for which $\psi = 0$.

If we draw lines through O parallel to the generators of this cone and then let these lines revolve as Q moves round the circle we do not get all lines through O but only those outside a certain right circular cone whose axis is perpendicular to the plane of the orbit.

The points in the plane of the orbit are of some interest. The curve

$$r(c - v\cos\theta) = R\sqrt{(c^2 + v^2)}$$

goes to infinity in the two directions for which

$$c - v\cos\theta = \sqrt{(c^2 - v^2)}$$

and approaches O up to a certain minimum distance b which is nearly $\frac{a}{\sqrt{(2)}}$, where a is the radius of the orbit. Hence the points in the plane of the orbit for which $\psi = 0$ lie outside the circle $R = b$.

If in a quantum jump from one circular orbit to a coplanar circular orbit energy is emitted in the form of an electric particle which travels with velocity c, this particle probably originates in the plane of the orbit.

Let O be the centre of mass of the atom before emission. After emission the centre of mass of the quantum and the centre of mass of the atom will ultimately be travelling along parallel lines whose distances from O are p and q, say. If the quantum has energy $h\nu$ and linear momentum $h\nu/c$, the atom will ultimately have a linear momentum $h\nu/c$ in the opposite direction.

Since the angular momentum about an axis through O perpendicular to the plane of the orbit remains unchanged we have the equation

$$\frac{h\nu}{c}(p + q) = \text{change in angular momentum of atom about its centre of mass}$$

$$= \frac{h}{2\pi}.$$

Therefore $$p + q = \frac{2\pi c}{\nu} = 2\pi\lambda.$$

This equation, which is obtained on the assumption that the quantum has no angular momentum about its centre of mass, seems to indicate that the quantum travels ultimately as if it did not come directly from the interior of the atom but from a point somewhere outside. This may mean, of course, that the quantum moves along a curve as it leaves the atom and approaches an asymptote which is at distance $2\pi\lambda$ from the final line of motion of the atom.

A PROOF APPLICABLE TO INTERSECTING SURFACES OF NOETHER'S FUNDAMENTAL THEOREM AS TO INTERSECTING CURVES.

By *E. B. Elliott.*

1. NOETHER'S theorem is that, if the nm intersections of two curves represented by homogeneous equations $S_m = 0$, $S_n = 0$, of degrees indicated by the suffixes, are distinct, and if another curve $S_p = 0$ $(p \geqq m \geqq n)$ passes through all these intersections, then an identity holds of the form

$$S_p \equiv A_{p-m} S_m + B_{p-n} S_n.$$

It is not proposed to deal here with his extensions of the theorem to cases when the intersections of S_m, S_n are not all distinct, and appropriate limitations are imposed on S_p.

The theorem is often stated with inadequate precision. Most frequently perhaps S_m, S_n, S_p are taken to be non-homogeneous polynomials in two coordinates x, y, and it is only said and proved that polynomials A, B providing the identity exist. The vital fact for purposes to which the theorem is to be applied, that the degrees of these polynomials need not exceed $p-m$ and $p-n$, is left unexpressed.

A proof which actually exhibits the identity will now be given; and the method will be found to be applicable when S_m, S_n, S_p are homogeneous, not in three, but in four, or indeed in any number of variables, so that theorems akin to Noether's in geometry of three (or more) dimensions are yielded.

2. Let the reference be to a triangle whose (y, z) vertex is not on the curve $S_n = 0$, so that in

$$S_n \equiv v_0 x^n + v_1 x^{n-1} + \ldots + v_n \ldots\ldots\ldots\ldots\ldots (A),$$

where every v_r is an r-ic in y, z, the constant v_0 is not zero.

Simple division of S_m, S_p, regarded as polynomials in x, by S_n gives us identities

$$S_m \equiv T_{m-n} S_n + u_{m-n+1} x^{n-1} + \ldots + u_m \equiv T_{m-n} S_n + \sigma_m, \text{ say, } \ldots (B),$$

$$S_p \equiv T'_{p-n} S_n + w_{p-n+1} x^{n-1} + \ldots + w_p \equiv T_{p-n} S_n + \sigma_p, \text{ say, } \ldots (C),$$

where σ_m, σ_p are of dimensions m, p respectively in x, y, z collectively, but of degrees neither of which exceeds $n-1$ in x alone.

The intersections of S_m, S_n are those of σ_m, S_n; and the intersections of S_p, S_n those of σ_p, S_n. The latter np intersections include the former nm, all of which are supposed distinct. We can have so taken our (y, z) origin that the

nm lines from it to these distinct points are themselves distinct. The equation of these nm lines is obtained by equating to zero the x-eliminant of σ_m and S_n. An expression for this x-eliminant is the "dialytic" determinant of the multipliers of x^{2n-2}, x^{2n-3}, ..., 1 in the expressions $x^{n-1}\sigma_m$, $x^{n-2}\sigma_m$, ..., σ_m, $x^{n-2}S_n$, $x^{n-3}S_n$, ..., S_n, which involve them linearly.

For shortness of writing (the argument being general) let us take $n=3$, so that

$$x^2\sigma_m \equiv u_{m-2}x^4 + u_{m-1}x^3 + u_m x^2 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$x\sigma_m \equiv u_{m-2}x^3 + u_{m-1}x^2 + u_m x \ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$\sigma_m \equiv u_{m-2}x^2 + u_{m-1}x + u_m \ldots\ldots\ldots\ldots\ldots\ldots (3),$$
$$xS_3 \equiv v_0x^4 + v_1x^3 + v_2x^2 + v_3x \ldots\ldots\ldots\ldots\ldots\ldots (4),$$
$$S_3 \equiv v_0x^3 + v_1x^2 + v_2x + v_3 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

and let us also have before us

$$\sigma_p \equiv w_{p-2}x^2 + w_{p-1}x + w_p \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

Let Δ be the dialytic determinant of (1) to (5). The known theory of elimination tells us that whenever $\Delta=0$ there is an x making (1) to (5) vanish. With one (y, z) satisfying $\Delta=0$ goes one x, by the safeguard adopted in the reference. This x, with the y, z, specifies one point on $S_m=0$ and $S_n=0$. Such a point, by our data, also lies on $\sigma_p=0$, so that its coordinates make (6) vanish as well as (1) to (5), and the y, z with which the x goes make zero all the determinants Δ_1, Δ_2, ..., Δ_5 of (6) and sets of four out of (1) to (5). Accordingly all these five (*i.e.* $2n-1$) determinants, homogeneous functions of y, z, have the $(y, z)^{3m}$ which has been called Δ, a $(y, z)^{3m}$ without repeated factors, for a common divisor.

Observe now that, from the identities (1) to (6), we must have identically, whatever x, y, z be,

$$\begin{vmatrix} x^2\sigma_m & u_{m-2} & u_{m-1} & u_m & & \\ x\sigma_m & & u_{m-2} & u_{m-1} & u_m & \\ \sigma_m & & & u_{m-2} & u_{m-1} & u_m \\ xS_3 & v_0 & v_1 & v_2 & v_3 & \\ S_3 & & v_0 & v_1 & v_2 & v_3 \\ \sigma_p & & & w_{p-2} & w_{p-1} & w_p \end{vmatrix} \equiv 0 \ldots\ldots (7),$$

which may be written

$$\Delta\sigma_p - \Delta_5S_3 + \Delta_4xS_3 - \Delta_3\sigma_m + \Delta_2x\sigma_m - \Delta_1x^2\sigma_m \equiv 0.$$

This is an identity of dimensions $3m+p$ (*i.e.* $nm+p$) throughout in x, y, z collectively. Also Δ, of dimensions $3m$, is a common divisor of Δ_5, Δ_4, ..., Δ_1. Dividing by it we have an identity

$$\sigma_p - (P - Qx)\, S_3 - (R - Sx + Tx^2)\, \sigma_m \equiv 0$$

of dimensions p throughout; and this, when in it we substitute for σ_p and σ_m from (B) and (C), produces the identity

$$S_p \equiv A_{p-m} S_m + B_{p-3} S_3,$$

which was to be shown to exist.

3. To start again, let us mean by S_m, S_n, S_p an m-ic, an n-ic, and a p-ic homogeneous in four variables x, y, z, ω, regarded as tetrahedral coordinates, instead of the three variables of plane geometry. We may have taken a point not on the surface $S_n = 0$ for vertex (y, z, ω) of the tetrahedron of reference, so that S_n has a non-vanishing x^n term. Write S_n, S_m, S_p as in (A), (B), (C), where now the v's, u's, and w's are homogeneous in y, z, ω. Also write down again (1) to (6) and the identity (7), giving the new meanings to the letters involved. As before we shall arrive at a final identity

$$S_p \equiv A_{p-m} S_m + B_{p-n} S_n,$$

provided the circumstances are such that the new Δ_1, Δ_2, ..., Δ_5 are all divisible by the new Δ.

The complete intersection of surfaces $S_m = 0$, $S_n = 0$ consists of a curve or curves. We confine attention to cases in which no curve occurs twice as a partial intersection, but such multiple points as there may be are isolated. Let it be given us that the complete intersection of $S_m = 0$, $S_n = 0$ is part (or the whole) of the complete intersection of $S_p = 0$, $S_n = 0$. We will see that under these circumstances the requirements at the end of the last paragraph are satisfied.

We can have taken our (y, z, ω) origin, not only off the surface $S_n = 0$, but also not on any line joining two points of the complete intersection of $S_m = 0$, $S_n = 0$. If now the new dialytic determinant Δ vanishes, there is an x going with any y, z, ω for which it vanishes, which makes the new (1) to (5) all vanish. Such an x, with the associated y, z, ω, specifies a point on $S_p = 0$ as well as on $S_m = 0$ and $S_n = 0$, and so makes the new (6) vanish. The y, z, ω therefore also make zero every one of the new Δ_1, Δ_2, ..., Δ_5. These therefore vanish whenever Δ does. Also Δ has no repeated factor; for we have excluded cases when there are repeated cones from the (y, z, ω) origin to loci forming parts of the complete

intersection of $S_m = 0$, $S_n = 0$. Therefore Δ is a common factor of $\Delta_1, \Delta_2, \ldots, \Delta_5$. Thus:

If, with $p \geq m \geq n$, a surface $S_p = 0$ passes through the complete intersection of two surfaces $S_m = 0$, $S_n = 0$, and if this complete intersection has no repeated part other than isolated multiple points, an identity $S_p \equiv A_{p-m} S_m + B_{p-n} S_n$ holds.

It follows that what is left of the complete intersection of S_p and S_n when we take away the complete intersection of S_m and S_n is the complete intersection of S_n and a $(p-m)$-ic surface A_{p-m}, and that what is left of the complete intersection of S_p and S_m when we take away the complete intersection of S_m and S_n is the complete intersection of S_m and a $(p-n)$-ic surface B_{p-n}.

Accordingly a theory of residual and co-residual loci on an n-ic surface, parallel to the familiar theory of residual and co-residual sets of points on an n-ic curve, can be developed, just as the latter has been developed by aid of Noether's theorem.

The surfaces in the parallel theory may be composite, just as the curves may be composite in the older theory.

4. As an example of the use of § 3 let us follow in three dimensions the steps of a familiar proof of Pascal's theorem and its converse by aid of Noether's. Let $L_1L_3L_5 = 0$ and $L_2L_4L_6 = 0$ be two sets of three planes. If three of their nine lines of intersection form the complete intersection of $L_1L_3L_5 = 0$ with a plane $M = 0$, there must be a quadric surface $Q = 0$ such that $L_2L_4L_6 \equiv kL_1L_2L_3 + QM$, so that the six of the lines of intersection of $L_2L_4L_6$ and $L_1L_3L_5$ which do not lie on $M = 0$ must lie on this quadric $Q = 0$; and again, if what we are given is that six of the nine lines of intersection are generators of a quadric $Q = 0$, there must be a plane $M = 0$ containing the three other lines of intersection. Give the name (rs) to the line of intersection of $L_r = 0$ and $L_s = 0$. The lines (12), (23), (34), (45), (56), (61) are the sides of a skew hexagon in space $ABCDEF$, and the lines (14), (25), (36) are the lines of intersection of the planes FAB, CDE, the planes ABC, DEF, and the planes BCD, EFA; so that we get two theorems: (1) If $ABCDEF$ is a skew hexagon, and the planes FAB, ABC, BCD intersect the planes CDE, DEF, EFA respectively in coplanar lines, then the sides of the hexagon are generators of a quadric; and conversely (2) if the sides of a skew hexagon are generators of a quadric surface, the tangent planes to the surface at pairs of opposite vertices intersect in the three coplanar lines.

ON AN EXTENSION OF MILNE'S INTEGRAL EQUATION.

By *B. M. Wilson.*

It has been shown by A. Milne that, if we denote by

(1) $$D_n(x) = (-1)^n e^{\frac{1}{4}x^2} \frac{d^n}{dx^n}(e^{-\frac{1}{2}x^2}),$$

Weber's functions (of integral order) associated with the parabolic cylinder, then these functions satisfy the homogeneous integral equations*

(2) $$\int_0^\infty D_{2n}(y) \cos(\tfrac{1}{2}xy)\, dy = (-1)^n \sqrt{\pi}\, D_{2n}(x),$$

(3) $$\int_0^\infty D_{2n+1}(y) \sin(\tfrac{1}{2}xy)\, dy = (-1)^n \sqrt{\pi}\, D_{2n+1}(x).$$

It is well known that Sonine's polynomial $T_\nu^n(x)$, which is defined by the equation†

(4) $$T_\nu^n(x) = \frac{(-1)^n e^x x^{-\nu}}{n!\, \Gamma(n+\nu+1)} \frac{d^n}{dx^n}(e^{-x} x^{n+\nu})$$
$$= \sum_{k=0}^{n} \frac{(-1)^k x^{n-k}}{k!\,(n-k)!\,\Gamma(n+\nu-k+1)},$$

affords a ready generalisation of the function $D_n(x)$.

In fact, writing

(5) $$U_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})\ddagger$$
$$= 2^{\frac{1}{2}n} e^{\frac{1}{2}x^2} D_n(x\sqrt{2}) = n! \sum_{k=0}^{\frac{1}{2}[n]} \frac{(-1)^k x^{n-2k}}{k!\,(n-2k)!},$$

we have

(6) $$T_{-\frac{1}{2}}^n(x^2) = \frac{1}{\sqrt{\pi}} \frac{U_{2n}(x)}{(2n)!},$$

(7) $$x\, T_{\frac{1}{2}}^n(x^2) = \frac{1}{\sqrt{\pi}} \frac{U_{2n+1}(x)}{(2n+1)!}.$$

It is therefore seen that equations (2) and (3) are contained as special cases, for $\nu = -\frac{1}{2}$ and $\nu = \frac{1}{2}$, of the

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* A. Milne, *Proc. Edin. Math. Soc.*, vol. xxxii. (1914), pp. 2–14.

† See *e.g.* Gegenbauer, *Wien. Sitzungsber.* vol. xcv. (1887), pp. 274–290; or Bateman, *Electrical and Optical Wave Motion*, pp. 99–100.

‡ The polynomial $U_n(x)$ is, but for the factor $(-1)^n$, identical with Hermite's polynomial. See Hermite, *Comptes Rendus*, vol. lviii. (1864), p. 93; or *Oeuvres Complètes*, vol. ii., p. 291 *et seq.*

following theorem, a proof of which will now be given: *The integral equation*

$$(8) \qquad f(x) = \lambda \int_0^\infty \sqrt{(xy)}\, J_\nu(xy) f(y)\, dy,$$

wherein $R(\nu) > -1$, *has characteristic numbers* $\lambda = \pm 1$, *and the corresponding solutions are*

$$(9) \qquad f(x) = e^{-\frac{1}{2}x^2} x^{\nu+\frac{1}{2}} T_\nu^n(x^2), \quad (n = 0, 1, 2, \ldots).$$

To prove this we will establish the relation

$$(10) \quad \tfrac{1}{2}\int_0^\infty e^{-\frac{1}{2}x} x^{\frac{1}{2}\nu} T_\nu^n(x)\, J_\nu\{\sqrt{(xy)}\}\, dx = (-1)^n e^{-\frac{1}{2}y} y^{\frac{1}{2}\nu} T_\nu^n(y), \quad \{R(\nu) > -1\}^* ;$$

for on replacing in this equation x by x^2 and y by y^2 we obtain the desired result. Write

$$A_n = \tfrac{1}{2}\int_0^\infty e^{-\frac{1}{2}x} x^{\frac{1}{2}\nu} T_\nu^n(\tfrac{1}{2}x)\, J_\nu\{\sqrt{(xy)}\}\, dx$$

$$= 2^{\frac{1}{2}\nu}\int_0^\infty e^{-x} x^{\frac{1}{2}\nu} T_\nu^n(x)\, J_\nu\{\sqrt{(2xy)}\}\, dx$$

$$= \frac{(-1)^n 2^{\frac{1}{2}\nu}}{n!\,\Gamma(n+\nu+1)}\int_0^\infty x^{-\frac{1}{2}\nu} J_\nu\{\sqrt{(2xy)}\}\frac{d^n}{dx^n}(e^{-x}x^{n+\nu})\, dx, \quad \{R(\nu) > -1\},$$

in virtue of (4). Hence, integrating n times by parts, and noting that the integrated terms all vanish at both limits and that

$$\frac{d^n}{dx^n}\left[x^{-\frac{1}{2}\nu} J_\nu(\sqrt{x})\right] = (-\tfrac{1}{2})^n x^{-\frac{1}{2}(n+\nu)} J_{n+\nu}(\sqrt{x}),$$

we find that

$$A_n = \frac{(-1)^n 2^{\frac{1}{2}(\nu-n)} y^{\frac{1}{2}n}}{n!\,\Gamma(n+\nu+1)}\int_0^\infty e^{-x} x^{\frac{1}{2}(n+\nu)} J_{n+\nu}\{\sqrt{(2xy)}\}\, dx$$

$$= \frac{(-\frac{1}{2})^n y^{\frac{1}{2}n}}{n!\,\Gamma(n+\nu+1)}\int_0^\infty e^{-\frac{1}{2}t^2} t^{n+\nu+1} J_{n+\nu}(t\sqrt{y})\, dt, \quad (2x = t^2),$$

$$= \frac{(-\frac{1}{2})^n y^{n+\frac{1}{2}\nu} e^{-\frac{1}{2}y}}{n!\,\Gamma(n+\nu+1)} \dagger$$

in virtue of a special case of an integral evaluated by Hankel.‡

* The assumption that $R(\nu) > -1$ is, of course, necessary in order to ensure the convergence of the integral at its lower limit.

† A result equivalent to this evaluation of A_n has been given by Gegenbauer. *loc. cit.*, p. 278.

‡ See *e.g.* Watson, *Theory of Bessel Functions*, p. 391. It may be noted, in passing, that the result here made use of is equation (10) itself with $n = 0$.

Now from the series for $T_\nu^n(x)$ given in (4) it is readily deduced that

$$T_\nu^n(x) = \sum_{k=0}^{n} \frac{2^{n-k}}{k\,!}\, T_\nu^{(n-k)}(\tfrac{1}{2}x);$$

consequently

$$\tfrac{1}{2}\int_0^\infty e^{-\frac{1}{2}x}\, x^{\frac{1}{2}\nu}\, T_\nu^n(x)\, J_\nu\{\sqrt{(xy)}\}\, dx = \sum_{k=0}^{n} \frac{2^{n-k}}{k\,!} A_{n-k}$$
$$= (-1)^n e^{-\frac{1}{2}y}\, y^{\frac{1}{2}\nu}\, T_\nu^n(y),$$

on substituting the expression found for A_{n-k} and comparing with the series in (4). Thus the result enunciated is proved.

Equation (10) has obvious formal relations with Hankel's representation of an "arbitrary" function by a double integral* and with the representation of the function by means of its "Fourier-Sonine" series, namely

$$f(x) = \sum_{n=0}^{\infty} a_n\, e^{-\frac{1}{2}x}\, x^{\frac{1}{2}\nu}\, T_\nu^n(x),$$

where

$$a_n = n\,!\,\Gamma(n+\nu+1)\int_0^\infty e^{-\frac{1}{2}x}\, x^{\frac{1}{2}\nu} f(x)\, T_\nu^n(x)\, dx.$$

It is possibly worth noticing in conclusion why, in addition to the analogy with Fourier's Integral Equation (2) or (3), and in addition to arguments drawn from Hankel's Inversion Formula, the nucleus of the homogeneous equation (8) is taken (with apparent arbitrariness) as $\sqrt{(xy)}\, J_\nu(xy)$; and no attempt is made to consider the similar equation with nucleus

$$(xy)^{k+\frac{1}{2}} J_\nu(xy), \quad (k \neq 0).$$

For it may readily be shown that if three functions f, ϕ and ψ are connected by the relation

$$(11) \qquad \int_0^\infty f(x)\, \phi(xy)\, dx = \psi(y), \quad (y \geq 0),$$

and if

$$\int_0^\infty x^{s-1}\,|\,\phi(x)\,|\,dx, \quad \int_0^\infty x^{-s}\,|f(x)|\,dx, \quad \int_0^\infty x^{s-1}\psi(x)\,dx$$

all exist for $\qquad \alpha \leq R(s) \leq \beta,$

then

$$(12) \qquad F(1-s)\,\Phi(s) \equiv \Psi(s),$$

where $\qquad F(s) = \int_0^\infty x^{s-1} f(x)\, dx,$

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* Watson, *loc. cit*, pp. 456–464.

with similar equations for Φ and Ψ. But, for the homogeneous equation,

$$f(x) \equiv \lambda\psi(x), \quad F(s) = \lambda\Psi(s),$$

and therefore (12) becomes

$$\lambda F(1-s)\,\Phi(s) = F(s). \tag{13}$$

On replacing s by $1-s$ in (13) and multiplying the two equations we obtain, as a necessary condition for the solubility of the equation by functions with the prescribed properties,

$$\lambda^2\Phi(s)\,\Phi(1-s) = 1. \tag{14}$$

But, if

$$\phi(x) = x^{k+\frac{1}{2}} J_\nu(x),$$

$$\Phi(s) = \int_0^\infty x^{s+k-\frac{1}{2}} J_\nu(x)\,dx$$

$$= 2^{s+k-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s + \frac{1}{2}k + \frac{1}{4})}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s - \frac{1}{2}k + \frac{3}{4})}, \text{*}$$

provided that

$$-\tfrac{1}{2} - R(\nu+k) < R(s) < 1 - R(k).$$

From this expression it is at once verified that $\Phi(s)\,\Phi(1-s)$ is independent of s if, and only if, k is zero.

Thus such solutions as may be possessed by the more general equation with $k \neq 0$ would necessarily be of a more recondite character than those of the equation here discussed. For example, when

$$-1 - R(\nu) < R(k) < \tfrac{1}{2},$$

$$\int_0^\infty x^k y^{k+\frac{1}{2}} J_\nu(xy)\,dx = \frac{1}{\sqrt{y}}\int_0^\infty u^k J_\nu(u)\,du = 2^k \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}k + \frac{1}{2})}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}k + \frac{1}{2})}\frac{1}{\sqrt{y}};$$

so that, for such values of k and ν,

$$f(x) \equiv \frac{1}{\sqrt{x}}$$

is a solution of the generalised equation, with

$$\lambda = 2^{-k}\frac{\Gamma(\frac{1}{2}\nu - \frac{1}{2}k + \frac{1}{2})}{\Gamma(\frac{1}{2}\nu + \frac{1}{2}k + \frac{1}{2})}.$$

But to obtain solutions other than this trivial one† would, no doubt, present considerable difficulty.

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* See Watson, *loc. cit.*, p. 391.

† Compare Humbert, *Proc. Edin. Math. Soc.*, vol. xxxii (1914), p. 22, for $\frac{1}{\sqrt{x}}$ as a solution of equations of the type considered.

ON A PARTIAL LINEAR DIFFERENCE EQUATION.

By *Prof. W. Burnside.*

Let $f(m_1, m_2, \ldots, m_s)$ denote a function of s integers m_1, $m_2, \ldots, m_s$ positive, negative, or zero. It is proposed to consider those solutions of the difference equation

$$\sum_1^s \{f(m_1, m_2, \ldots, m_i - 1, m_{i+1}, \ldots, m_s)$$

$$+f(m_1, m_2, \ldots, m_i + 1, m_{i+1}, \ldots, m_s)\} - 2sf(m_1, m_2, \ldots, m_s) = 0,$$

in which $f(m_1, m_2, \ldots, m_s)$ approaches zero as $|m_i|$ approaches infinity for any value of i.

If the equation is satisfied for every set of s integers, and if $f(m_1, m_2, \ldots, m_s)$ is not a mere constant, then at least one of the differences

$$f(m_1, m_2, \ldots, m_i \pm 1, m_{i+1}, \ldots, m_s) - f(m_1, m_2, \ldots, m_s)$$

$$(i = 1, 2, \ldots, s)$$

must be positive, so that it is possible to pass continually from a set of integers to an "adjacent" set in such a way that f increases. This is clearly impossible if $f(m_1, m_2, \ldots, m_s)$ approaches zero when $|m_i|$ approaches infinity for any i: and therefore a solution of the difference equation, which satisfies this condition, must fail to satisfy the equation for at least one set of s integers.

Lemma I. Let $m_1, m_2, \ldots, m_s$ be s positive integers, any one or more of which may be zero: and let $r_1, r_2, \ldots, r_s$ be s positive integers (or zeros), such that

$$\sum_1^s r_i = r.$$

Put
$$f = \frac{(2r + \sum_1^s m_i)!}{\prod_{i=1}^s r_i!\,(m_i + r_i)!},$$

and let $$F_r = \Sigma f.$$

where the sum is taken for all positive or zero values of $r_1, r_2, \ldots, r_s$. Then if $s > 2$

$$\sum_0^\infty \frac{F_r}{(2s)^{2r}}$$

is a convergent series.

Write $$r_i = \frac{r}{s} + t_i,$$

and in calculating $\log f$, use the known approximation for the factorials, assuming that r is large, viz.

$$\log n! = \tfrac{1}{2}\log 2\pi + (n+\tfrac{1}{2})\log n - n + \frac{1}{12n} + \ldots.$$

It will be found that, if $\sum\limits_1^s m_i = m$,

$$\log f = -\frac{2s-1}{2}\log 2\pi + (m + 2r + \tfrac{1}{2})\log 2 + (m + 2r + s)\log s$$

$$-(s-\tfrac{1}{2})\log r - \frac{s}{2r}\left[\sum_1^s t_i^2 + \sum_1^s (t_i + m_i)^2\right] + \frac{A}{r} + \text{terms in } \frac{1}{r^2},$$

where A is a finite constant, depending only on m and s.

Hence $$f = \frac{2^{m+\frac{1}{2}} s^{m+s}}{(2\pi)^{\frac{1}{2}(2s-1)}} (2s)^{2r} \frac{1}{r^{s-\frac{1}{2}}} e^{-\frac{s}{2r}\left[\sum\limits_1^s t_i^2 + \sum\limits_1^s (t_i+m_i)^2\right] + \frac{A}{r} + \frac{B}{r^2}}.$$

The factor $e^{\frac{A}{r}}$ rapidly approaches unity as r increases; and before the factor $e^{\frac{B}{r^2}}$ becomes appreciably different from unity, the factor $e^{-\frac{s}{2r}\left[\sum\limits_1^s t_2^2 + \sum\limits_1^s (t_2+m_2)^2\right]}$ becomes extremely small. Hence if r is sufficiently great, the leading term in f is

$$C\frac{(2s)^r}{r^{s-\frac{1}{2}}} e^{-\frac{s}{2r}\sum\limits_1^s (t_i + \frac{1}{2}m_i)^2},$$

where C is independent of r. The leading term in F_r is

$$C\frac{(2s)^r}{r^{s-\frac{1}{2}}} \Sigma e^{-\frac{s}{2r}\sum\limits_1^s t_2'^2}, \qquad t_i' = t_i + \tfrac{1}{2}m_i,$$

where the sum is to be taken for all values of the t''s increasing by unity at a step, which satisfy

$$t_i' \geq -\frac{r}{s} + \tfrac{1}{2}m_i, \qquad \sum_1^s t_i' = \tfrac{1}{2}m.$$

When r is large enough this sum is very nearly equal to the $(s-1)$-ple integral

$$\iint_{-\infty}^{\infty}\cdots\int e^{-\frac{s}{2r}(t_1^2+t_2^2+\ldots+t_{s-1}^2+(\frac{1}{2}m-t_1-t_2-\ldots-t_{s-1})^2)}\,dt_1dt_2\ldots dt_{s-1},$$

which is known to be of the form

$$C'r^{\frac{s-1}{2}},$$

where C' is a finite constant depending on m and s. Hence the leading term in F_r is

$$D\frac{(2s)^r}{r^{\frac{1}{2}s}},$$

and the series is, as stated, convergent if $s \geq 3$.

LEMMA II. Since, when r is great enough, the general term in $\Sigma \dfrac{F_r}{(2s)^r}$ is of the form $\dfrac{D}{r^{\frac{1}{2}s}}$; it follows that when $t > s \geq 3$, and t approaches s uniformly, then $\Sigma \dfrac{F_r}{(2t)^r}$ approaches $\Sigma \dfrac{F_r}{(2s)^r}$ uniformly.

If $t > s$, there is a range of real positive values of $x_1, x_2, \ldots, x_s$, for which

$$\sum_1^s (x_i + x_i^{-1}) < 2t.$$

For this range of values

$$\frac{1}{2t - \sum\limits_1^s (x_i + x_i^{-1})} = \sum_{n=0}^{\infty} \frac{\{\sum\limits_1^s (x_i + x_i^{-1})\}^n}{(2t)^{n+1}}.$$

Since every term is positive, the separate terms on the right may be expanded and re-arranged: and when this is done the coefficient of $x_1^{m_1}x_2^{m_2}, \ldots, x_s^{m_s}$ depends only on $|m_1|, |m_2|, \ldots, |m_s|$. Suppose that $m_1, m_2, \ldots, m_s$ are positive, and that their sum is m. Then a term in $x_1^{m_1}x_2^{m_2}\ldots x_s^{m_s}$ will occur in

$\{\sum_1^s (x_i + x_i^{-1})\}^n$ if $n = m + 2r$, where r is zero or a positive integer. Moreover, the coefficient of $x_1^{m_1}x_2^{m_2}\dots x_s^{m_s}$ in $\{\sum_1^s (x_i + x_i^{-1})\}^{m+2r}$ is

$$\Sigma \frac{(m+2r)!}{\prod_1^s r_i!\,(m_i + r_i)!},$$

where the sum is taken for all positive (or zero) values of $r_1, r_2, \dots, r_s$, such that

$$\sum_1^s r_i = r.$$

With the notation of Lemma I this is F_r.

Hence within the range of real positive values of the variables for which

$$\sum_1^s (x_i + x_i^{-1}) < 2t,$$

$$\frac{1}{2t - \sum_1^s (x_i + x_i^{-1})} = \sum_{m_1, m_2 \dots = -\infty}^{m_1, m_2 \dots = +\infty} \Sigma\dots\Sigma\; x_1^{m_1}x_2^{m_2}\dots x_s^{m_s} g'(|m_1|, |m_2|, \dots, |m_s|),$$

where $\qquad g'(|m_1|, |m_2|, \dots, |m_s|) = \sum_{r=0}^{\infty} \frac{F_r}{(2t)^{m+2r+1}}.$

Within the same range

$$1 \equiv [2t - \sum_1^s (x_i + x_i^{-1})][\sum_{-\infty}^{\infty}\dots\Sigma x_1^{m_1}x_2^{m_2}\dots x_s^{m_s} g'(|m_1|, |m_2|, \dots, |m_s|)]$$

is an identity. Hence

$$\sum_1^s \{g'(|m_1|, \dots, |m_i - 1|, \dots, |m_s|) + g'(|m_1|, \dots, |m_i + 1|, \dots, |m_s|)\}$$
$$- 2tg'(|m_1|, \dots, |m_i|, \dots, |m_s|) = 0$$

for all sets of integers $m_1, m_2, \dots, m_s$; except that the right-hand side is -1 for $m_i = 0$ $(i = 1, 2, \dots, s)$.

Hence, by Lemma II., if

$$g(|m_1|, |m_2|, \dots, |m_s|) = \sum_{r=0}^{\infty} \frac{F_r}{(2s)^{m+2r+1}},$$

$$f(m_1, m_2, \dots, m_s) = g(|m_1|, |m_2|, \dots, |m_s|)$$

is a solution of the original difference equation, which holds for all values of $m_1, m_2, \ldots, m_s$ except simultaneous zero values in which case the right-hand side is -1.

If $r_1, r_2, \ldots, r_s$ is a *particular* set of positive integers whose sum is r, then

$$\frac{(n+2r)!}{\prod\limits_1^s r_i!\,(m_i+r_i)!}\,\frac{1}{(2s)^{m+2r+1}} \text{ and } \frac{(n+2r)!}{\prod\limits^s r_i!\,(n_i+r_i)!}\,\frac{1}{(2s)^{n+2r+1}}$$

may be described as corresponding terms in

$$g(|m_1|, |m_2|, \ldots, |m_s|) \text{ and } g(|n_1|, |n_2|, \ldots, |n_s|).$$

The ratio of any term in $g(|m_1+1|, |m_2|, \ldots, |m_s|)$ to the corresponding terms in $g(|m_1|, |m_2|, \ldots, |m_s|)$ is then

$$\frac{m+1+2r}{m+2r}\,\frac{m_1+r_1}{m_1+1+r_1}\,\frac{1}{2s},$$

which is equal to or less than

$$\frac{m+1}{2ms},$$

so that
$$\frac{g(|m_1+1|, |m_2|, \ldots, |m_s|)}{g(|m_1|, |m_2|, \ldots, |m_s|)} \leq \frac{m+1}{2ms},$$

and therefore $g(|m_1|, |m_2|, \ldots, |m_s|)$ approaches zero as $|m_1|$ approaches infinity.

There can be no second solution satisfying the same conditions as $g(|m_1|, |m_2|, \ldots, |m_s|)$; for their difference would satisfy the difference equation for *every* set of s integers and would therefore necessarily be zero.

The most general solution of the difference equation of the type considered is

$$\Sigma C_p g(|m_1-p_1|, |m_2-p_2|, \ldots, |m_s-p_s|).$$

For this solution

$$\sum_1^s \{g(p_1, \ldots, p_i-1, \ldots, p_s) + g(p_1, \ldots, p_i+1, \ldots, p_s)\} - 2sg(p_1, \ldots, p_i, \ldots, p_s) = -C_p.$$

The case $s=2$, in which the series $\Sigma F_r/(2s)^{2r}$ is not convergent, has been treated already.*

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* *Proceedings of the Cambridge Philosophical Society*, vol. xxi., part 5.

THETA EXPANSIONS USEFUL IN ARITHMETIC.

By *E. T. Bell*, University of Washington.

As emphasized by Glaisher, the unique and arithmetically useful form of a theta expansion is that in which the coefficients of the several powers of q are given as functions of the divisors of the exponent. This type of development also occurs in Hermite's celebrated *Lettre à Liouville*, and there can be little doubt that Liouville himself made frequent use of such expansions, although, according to his custom, he suppressed the analysis by which he reached his results. Tedious reductions being frequently required to change the analytical expansions to their simplest arithmetical equivalents, it is desirable for use in the theory of numbers that the series be given directly in their proper form. Many of the more complicated developments being of only slight interest for other than arithmetical applications, they are not available in the standard treatises and papers.

The present paper is a first supplement to the memoirs of Beihler,* Humbert,† and Petr.‡

In the first two of these there are derived a large number of important expansions which, when reduced to arithmetical form, involve incomplete functions of divisors. Such a function of the divisors d of the positive integer n is one containing only those d's that satisfy an inequality, usually $d < \sqrt{n}$. All of the functions in the following lists are complete. As such lists are most useful when they contain all the series of a given type, we have included the reduced expansions of all functions that can be derived from others by transformations of the first and second orders.

Two immediate uses of these lists may be noted. It will be seen by inspection of §§ 3–22 that we have obtained the

* *Sur les developpements en séries des fonctions doublement périodiques de troisième espèce*, Thèse, Paris, 1879.

† "Formules relatives aux nombres de classes, etc.", *Journal des Mathématiques*, 6 série, vol. iii. (1907), pp. 337–449.

‡ Published in Bohemian, references in Dickson's *History of the Theory of Numbers*, vol. iii. (1923), chap. 6.

expansions of functions of the form $T_1 T_2 T_3 \ldots$, where the series for each of the theta quotients $T_1, T_2, T_3, \ldots$, is given in the lists. Similarly for the derivatives of the functions. Hence the method of paraphrase may be applied directly to read off from the lists a great many general arithmetical theorems concerning wholly arbitrary odd or even functions. The second application applies the same method to identities between series taken from the following lists and those of Biehler, Humbert, and Petr. This gives a prolific source of new and generalized binary quadratic class-number relations for both definite and indefinite forms.

1. In all the series the outer Σ refers to the positive non-zero integer m, n, α or β in the exponent of q, and extends to all $m = 1, 3, 5, \ldots$, to all $n = 1, 2, 3, \ldots$, to all $\alpha = 1, 5, 9, \ldots$, to all $\beta = 3, 7, 11, \ldots$. The coefficient of the power of q is in [ ] after the power. The letters t, τ, d, δ denote positive integral divisors, and τ *is always odd.* The inner Σ refers to all the indicated divisors t, τ, d, δ of the m, n, α or β in the exponent of q, such that $m = t\tau$, $n = t\tau = d\delta$, $\alpha = t\tau$, $\beta = t\tau$. Note particularly that when the exponent of q is cm, cn, $c\alpha$ or $c\beta$, where c is a numerical constant, the inner Σ refers to the divisors of m, n, α or β, *not* to those of cm, cn, $c\alpha$ or $c\beta$. From the notation it follows that t in $n = t\tau$ is any divisor of n which is of the same parity as n; in $m = t\tau$, $\alpha = t\tau$, $\beta = t\tau$, t is odd since m, α, β, τ are odd; in $n = d\delta$ either of d, δ may be odd or even if n is even; and in $\alpha = t\tau$, $t \equiv \tau \bmod 4$, while in $\beta = t\tau$, $t \equiv -\tau \bmod 4$, since $\alpha \equiv 1 \bmod 4$, $\beta \equiv -1 \bmod 4$.

To simplify the printing we have used Jacobi's $(a \mid b)$,

$$(-1 \mid m) = (-1)^{(m-1)/2}, \quad (2 \mid m) = (-1)^{(m^2-1)/8}.$$

The small theta notation of Jacobi,* with $\vartheta_0(x)$ for his $\vartheta(x)$, is used. In this notation the argument of the circular functions does not, as in some others, contain the factor π.

The letters x', y' denote respectively $x + \pi/2$, $y + \pi/2$.

2. For the following 16 values of the triple index abc,

001,	010,	023,	032,
100,	111,	122,	133,
203,	212,	221,	230,
302,	313,	320,	331,

* *Werke*, vol. i, p. 501.

we write the doubly periodic functions of the second kind in the form

$$\phi_{abc}(x, y) = \frac{\vartheta_1'\vartheta_a(x+y)}{\vartheta_b(x)\vartheta_c(y)}.$$

Hence $\phi_{abc}(x, y) = \phi_{acb}(y, x)$, and we have

$$\phi_{203}(x, y) = \phi_{100}(x, y'), \quad \phi_{133}(x, y) = -\phi_{100}(x', y');$$

$$\phi_{302}(x, y) = \phi_{001}(x, y'), \quad \phi_{331}(x, y) = \phi_{001}(x', y),$$

$$\phi_{032}(x, y) = \phi_{001}(x', y');$$

$$\phi_{212}(x, y) = \phi_{111}(x, y'), \quad \phi_{122}(x, y) = -\phi_{111}(x', y').$$

By changes of x, y into x', y' and of (x, y) into (y, x) we therefore obtain all 16 expansions from

$$\phi_{100}(x, y) = 4\Sigma q^{m/2}[\Sigma \sin(tx + \tau y)],$$

$$\phi_{001}(x, y) = \csc y + 4\Sigma q^n[\Sigma \sin(2tx + \tau y)],$$

$$\phi_{111}(x, y) = \cot x + \cot y + 4\Sigma q^{2n}[\Sigma \sin 2(dx + \delta y)].$$

These are sufficient for reference here. The complete set is given in a former paper.† For rapid checking of the following lists the other paper should be used.

We shall write $D \equiv d/dz$, and denote by $D_x\phi_{abc}(u, v)$ the result of replacing (x, y) by (u, v) in $D_x\phi_{abc}(x, y)$, and similarly for the y derivative. The result of replacing z by zero in $D_z\vartheta_a(z) \equiv \vartheta_a'(z)$ is written ϑ_a'; $\vartheta_1(0) = 0$ and $\vartheta_a' \neq 0$ only when $a = 1$; $\vartheta_1' = \vartheta_0\vartheta_2\vartheta_3$.

3. The 12 finite functions $D_x\phi_{abc}(0, x)$ are obtained at once from § 2:

$$\vartheta_2\vartheta_3\vartheta_1'(x)/\vartheta_0(x) = 4\Sigma q^{m/2}[\Sigma\tau \sin tx],$$

$$\vartheta_0\vartheta_2\vartheta_2'(x)/\vartheta_0(x) = -4\Sigma q^{m/2}[\Sigma(-1|\tau)\tau \sin tx],$$

$$\vartheta_0\vartheta_3\vartheta_3'(x)/\vartheta_0(x) = -4\Sigma q^n[\Sigma(-1|\tau)\tau \cos 2tx];$$

$$\vartheta_2\vartheta_3\vartheta_0'(x)/\vartheta_1(x) = 8\Sigma q^n[\Sigma t \cos \tau x],$$

$$\vartheta_0\vartheta_3\vartheta_2'(x)/\vartheta_1(x) = -1 + 8\Sigma q^{2n}[\Sigma(-1)^d d \cos 2\delta x],$$

$$\vartheta_0\vartheta_2\vartheta_3'(x)/\vartheta_1(x) = 8\Sigma q^n[(-1)^n \Sigma t \cos \tau x];$$

$$\vartheta_0\vartheta_2\vartheta_0'(x)/\vartheta_2(x) = -8\Sigma q^n[(-1)^n \Sigma(-1|\tau) t \sin \tau x],$$

† *Messenger of Mathematics*, no. 581. vol. xlix. (1919), p. 83

$$\vartheta_0\vartheta_3\vartheta_1'(x)/\vartheta_2(x) = \quad 1-8\Sigma q^{2n}\left[\Sigma(-1)^{d+\delta}d\cos 2\delta x\right],$$
$$\vartheta_2\vartheta_3\vartheta_3'(x)/\vartheta_2(x) = -8\Sigma q^n\left[\Sigma(-1\,|\,\tau)\,t\sin\tau x\right];$$

$$\vartheta_0\vartheta_3\vartheta_0'(x)/\vartheta_3(x) = -4\Sigma q^n\left[(-1)^n\Sigma(-1\,|\,\tau)\,\tau\sin 2tx\right],$$
$$\vartheta_0\vartheta_2\vartheta_1'(x)/\vartheta_3(x) = \quad 4\Sigma q^{m/2}\left[(-1\,|\,m)\;\Sigma\tau\cos tx\right],$$
$$\vartheta_2\vartheta_3\vartheta_2'(x)/\vartheta_3(x) = \quad 4\Sigma q^{m/2}\left[\Sigma(-1\,|\,\tau)\,\tau\sin tx\right].$$

4. If in the expansion of $\phi_{abc}(x, y)$ as a power series in y we put $y=0$, we obtain the logarithmic derivatives of the thetas:

$$\vartheta_0'(x)/\vartheta_0(x) = \quad 4\Sigma q^n\left[\Sigma\sin 2tx\right],$$
$$\vartheta_1'(x)/\vartheta_1(x) = \quad \cot x + 4\Sigma q^{2n}\left[\Sigma\sin 2dx\right],$$
$$\vartheta_2'(x)/\vartheta_2(x) = -\tan x + 4\Sigma q^{2n}\left[\Sigma(-1)^d\sin 2dx\right],$$
$$\vartheta_3'(x)/\vartheta_3(x) = \quad 4\Sigma q^n\left[(-1)^n\Sigma\sin 2tx\right].$$

5. From $[D_x - D_y]\,\phi_{abc}(0, x)$ we get

$$\vartheta_2\vartheta_3\vartheta_1(x)\vartheta_0'(x)/\vartheta_0^2(x) = \quad 4\Sigma q^{m/2}\left[\Sigma(\tau-t)\cos tx\right],$$
$$\vartheta_0\vartheta_2\vartheta_2(x)\vartheta_0'(x)/\vartheta_0^2(x) = \quad 4\Sigma q^{m/2}\left[\Sigma(-1\,|\,\tau)(t-\tau)\sin tx\right],$$
$$\vartheta_0\vartheta_3\vartheta_3(x)\vartheta_0'(x)/\vartheta_0^2(x) = \quad 4\Sigma q^n\left[\Sigma(-1\,|\,\tau)(2t-\tau)\sin 2tx\right];$$

$$\vartheta_2\vartheta_3\vartheta_0(x)\vartheta_1'(x)/\vartheta_1^2(x) = \quad \csc x\cot x + 4\Sigma q^n\left[\Sigma(2t-\tau)\cos\tau x\right],$$
$$\vartheta_0\vartheta_3\vartheta_2(x)\vartheta_1'(x)/\vartheta_1^2(x) = \quad \cot^2 x - 8\Sigma q^n\left[\Sigma(-1)^\delta(d-\delta)\cos 2dx\right],$$
$$\vartheta_0\vartheta_2\vartheta_3(x)\vartheta_1'(x)/\vartheta_1^2(x) = \quad \csc x\cot x + 4\Sigma q^n\left[(-1)^n\Sigma(2t-\tau)\cos\tau x\right];$$

$$\vartheta_0\vartheta_2\vartheta_0(x)\vartheta_2'(x)/\vartheta_2^2(x) = -\sec x\tan x$$
$$+\,4\Sigma q^n\left[(-1)^n\Sigma(-1\,|\,\tau)(\tau-2t)\sin\tau x\right],$$
$$\vartheta_0\vartheta_3\vartheta_1(x)\vartheta_2'(x)/\vartheta_2^2(x) = -\tan^2 x + 8\Sigma q^{2n}\left[\Sigma(-1)^{d+\delta}(d-\delta)\cos 2dx\right],$$
$$\vartheta_2\vartheta_3\vartheta_3(x)\vartheta_2'(x)/\vartheta_2^2(x) = -\sec x\tan x$$
$$+\,4\Sigma q^n\left[\Sigma(-1\,|\,\tau)(\tau-2t)\sin\tau x\right];$$

$$\vartheta_0\vartheta_3\vartheta_0(x)\vartheta_3'(x)/\vartheta_3^2(x) = \quad 4\Sigma q^n\left[(-1)^n\Sigma(-1\,|\,\tau)(2t-\tau)\sin 2tx\right],$$
$$\vartheta_0\vartheta_2\vartheta_1(x)\vartheta_3'(x)/\vartheta_3^2(x) = \quad 4\Sigma q^{m/2}\left[(-1\,|\,m)\;\Sigma(\tau-t)\cos tx\right],$$
$$\vartheta_2\vartheta_3\vartheta_2(x)\vartheta_3'(x)/\vartheta_3^2(x) = \quad 4\Sigma q^{m/2}\left[\Sigma(-1\,|\,\tau)(t-\tau)\sin tx\right].$$

6. The $\phi_{abc}(x, -x)$ give

$$\vartheta_1'\vartheta_0/\vartheta_0(x)\vartheta_1(x) = \csc x + 4\Sigma q^n[\Sigma\sin(\tau-2t)x],$$
$$\vartheta_1'\vartheta_3/\vartheta_0(x)\vartheta_2(x) = \sec x + 4\Sigma q^n[\Sigma(-1|\tau)\cos(2t-\tau)x],$$
$$\vartheta_1'\vartheta_2/\vartheta_0(x)\vartheta_3(x) = 4\Sigma q^{m/2}[\Sigma(-1|\tau)\cos(t-\tau)x],$$
$$\vartheta_1'\vartheta_2/\vartheta_1(x)\vartheta_2(x) = 2\csc 2x + 4\Sigma q^{2n}[\Sigma(-1)^\delta\sin 2(d-\delta)x],$$
$$\vartheta_1'\vartheta_3/\vartheta_1(x)\vartheta_3(x) = \csc x + 4\Sigma q^n[(-1)^n\Sigma(\tau-2t)x],$$
$$\vartheta_1'\vartheta_0/\vartheta_2(x)\vartheta_3(x) = \sec x + 4\Sigma q^n[(-1)^n\Sigma(-1|\tau)\cos(\tau-2t)x].$$

The derivative of each of these is the sum of two in § 7.

7. The twelve finite functions $D_x\phi_{abc}(x, -x)$ give

$$\vartheta_1'\vartheta_0\vartheta_0'(x)/\vartheta_0^2(x)\vartheta_1(x) = 8\Sigma q^n[\Sigma t\cos(2t-\tau)x],$$
$$\vartheta_1'\vartheta_3\vartheta_0'(x)/\vartheta_0^2(x)\vartheta_2(x) = 8\Sigma q^n[\Sigma(-1|\tau)t\sin(2t-\tau)x],$$
$$\vartheta_1'\vartheta_2\vartheta_0'(x)/\vartheta_0^2(x)\vartheta_3(x) = 4\Sigma q^{m/2}[\Sigma(-1|\tau)t\sin(t-\tau)x];$$

$$\vartheta_1'\vartheta_0\vartheta_1'(x)/\vartheta_1^2(x)\vartheta_0(x) = \csc x\cot x - 4\Sigma q^n[\Sigma\tau\cos(2t-\tau)x],$$
$$\vartheta_1'\vartheta_2\vartheta_1'(x)/\vartheta_1^2(x)\vartheta_2(x) = \csc^2 x - 8\Sigma q^{2n}[\Sigma(-1)^\delta d\cos 2(d-\delta)x],$$
$$\vartheta_1'\vartheta_3\vartheta_1'(x)/\vartheta_1^2(x)\vartheta_3(x) = \csc x\cot x - 4\Sigma q^n[(-1)^n\Sigma\tau\cos(2t-\tau)x];$$

$$\vartheta_1'\vartheta_3\vartheta_2'(x)/\vartheta_2^2(x)\vartheta_0(x) = -\sec x\tan x$$
$$-4\Sigma q^n[\Sigma(-1|\tau)\tau\sin(2t-\tau)x],$$
$$\vartheta_1'\vartheta_2\vartheta_2'(x)/\vartheta_2^2(x)\vartheta_1(x) = -\sec^2 x + 8\Sigma q^{2n}[\Sigma(-1)^d d\cos 2(d-\delta)x],$$
$$\vartheta_1'\vartheta_0\vartheta_2'(x)/\vartheta_2^2(x)\vartheta_3(x) = -\sec x\tan x$$
$$-4\Sigma q^n[(-1)^n\Sigma(-1|\tau)\tau\sin(2t-\tau)x];$$

$$\vartheta_1'\vartheta_2\vartheta_3'(x)/\vartheta_3^2(x)\vartheta_0(x) = 4\Sigma q^{m/2}[\Sigma(-1|t)t\sin(t-\tau)x],$$
$$\vartheta_1'\vartheta_3\vartheta_3'(x)/\vartheta_3^2(x)\vartheta_1(x) = 8\Sigma q^n[(-1)^n\Sigma t\cos(2t-\tau)x],$$
$$\vartheta_1'\vartheta_0\vartheta_3'(x)/\vartheta_3^2(x)\vartheta_2(x) = 8\Sigma q^n[(-1)^n\Sigma(-1|\tau)t\sin(2t-\tau)x].$$

8. By the transformation of the second order we find from the third and first respectively of § 6 the first and second of the following. The third and fourth are then obtained from the second and first respectively by changing x into x' and reducing.

$$\vartheta_1'/\vartheta_0(x) = 2\Sigma q^{a/4}[\Sigma(-1|\tau)\cos\{\tfrac{1}{2}(t-\tau)x\}],$$
$$\vartheta_1'/\vartheta_1(x) = \csc x - 4\Sigma q^{2n}[\Sigma\sin(2t-\tau)x],$$

$$\vartheta_1'/\vartheta_2(x) = \sec x + 4\Sigma q^{2n}\left[(-1)^n \Sigma\,(-1\,|\,\tau)\cos(2t-\tau)\,x\right],$$
$$\vartheta_1'/\vartheta_3(x) = 2\Sigma q^{\alpha/4}\left[(2\,|\,\alpha)\,\Sigma\,(-1\,|\,\tau)\cos\{\tfrac{1}{2}(t-\tau)\,x\}\right].$$

9. The derivatives of those in § 8 are

$$\vartheta_1'\vartheta_0'(x)/\vartheta_0^2(x) = \Sigma q^{\alpha/4}\left[\Sigma\,(-1\,|\,\tau)\,(t-\tau)\sin\{\tfrac{1}{2}(t-\tau)\,x\}\right],$$
$$\vartheta_1'\vartheta_1'(x)/\vartheta_1^2(x) = \csc x\cot x + 4\Sigma q^{2n}\left[\Sigma\,(2t-\tau)\cos(2t-\tau)\,x\right],$$
$$\vartheta_1'\vartheta_2'(x)/\vartheta_2^2(x) = -\sec x\tan x$$
$$+\,4\Sigma q^{2n}\left[(-1)^n\Sigma\,(-1\,|\,\tau)\,(2t-\tau)\sin(2t-\tau)\,x\right],$$
$$\vartheta_1'\vartheta_3'(x)/\vartheta_3^2(x) = \Sigma q^{\alpha/4}\left[(2\,|\,\alpha)\,\Sigma\,(-1\,|\,\tau)\,(t-\tau)\sin\{\tfrac{1}{2}(t-\tau)\}\,x\right].$$

10. From $D_x\phi_{\alpha\beta\gamma}(x, -x)$ we get

$$\vartheta_1'^2/\vartheta_0^2(x) = 4\Sigma q^{m/2}\left[\Sigma t\cos(t-\tau)\,x\right],$$
$$\vartheta_1'^2/\vartheta_1^2(x) = \csc^2 x - 8\Sigma q^{2n}\left[\Sigma d\cos 2\,(d-\delta)\,x\right],$$
$$\vartheta_1'^2/\vartheta_2^2(x) = \sec^2 x - 8\Sigma q^{2n}\left[\Sigma\,(-1)^{d+\delta} d\cos 2\,(d-\delta)\,x\right],$$
$$\vartheta_1'^2/\vartheta_3^2(x) = 4\Sigma q^{m/2}\left[(-1\,|\,m)\,\Sigma t\cos(t-\tau)\,x\right].$$

11. The derivatives of those in § 10 give

$$\vartheta_1'^2\vartheta_0'(x)/\vartheta_0^3(x) = 2\Sigma q^{m/2}\left[\Sigma\,(d^2-m)\sin(d-\delta)\,x\right],$$
$$\vartheta_1'^2\vartheta_1'(x)/\vartheta_1^3(x) = \csc^2 x\cot x - 8\Sigma q^{2n}\left[\Sigma\,(d^2-n)\sin 2\,(d-\delta)\,x\right],$$
$$\vartheta_1'^2\vartheta_2'(x)/\vartheta_2^3(x) = -\sec^2 x\tan x$$
$$-\,8\Sigma q^{2n}\,\Sigma\left[(-1)^{d+\delta}(d^2-n)\sin 2\,(d-\delta)\,x\right],$$
$$\vartheta_1'^2\vartheta_3'(x)/\vartheta_3^3(x) = 2\Sigma q^{m/2}\left[(-1\,|\,m)\,\Sigma\,(d^2-m)\sin(d-\delta)\,x\right].$$

12. From § 10, by the transformation of the second order and by changes of q into $-q$ in the results, we find

$$\vartheta_1'^2\vartheta_0^2/\vartheta_0^2(x)\vartheta_1^2(x) = \csc^2 x - 8\Sigma q^n\left[\Sigma d\cos 2\,(d-\delta)\,x\right],$$
$$\vartheta_1'^2\vartheta_3^2/\vartheta_0^2(x)\vartheta_2^2(x) = \sec^2 x - 8\Sigma q^n\left[(-1)^n\Sigma(-1)^{d+\delta} d\cos 2\,(d-\delta)\,x\right],$$
$$\vartheta_1'^2\vartheta_2^2/\vartheta_0^2(x)\vartheta_3^2(x) = 16\,\Sigma q^m\left[\Sigma t\cos 2\,(t-\tau)\,x\right],$$
$$\vartheta_1'^2\vartheta_2^3/\vartheta_1^2(x)\vartheta_2^2(x) = 4\csc^2 2x - 32\Sigma q^{4n}\left[\Sigma d\cos 4\,(d-\delta)\,x\right],$$
$$\vartheta_1'^2\vartheta_3^2/\vartheta_1^2(x)\vartheta_3^2(x) = \csc^2 x - 8\Sigma q^n\left[(-1)^n\,\Sigma d\cos 2\,(d-\delta)\,x\right],$$
$$\vartheta_1'^2\vartheta_0^2/\vartheta_2^2(x)\vartheta_3^2(x) = \sec^2 x - 8\Sigma q^n\left[\Sigma\,(-1)^{d+\delta} d\cos 2\,(d-\delta)\,x\right].$$

13. Taking the differences of those pairs of fractions in § 7 which have the same two theta functions in their denomi-

nators, and reducing by means of the differential relations between the thetas, we find

$$\vartheta_1'\vartheta_0^3\vartheta_2(x)\vartheta_3(x)/\vartheta_0^2(x)\vartheta_1^2(x) = \cot x \csc x$$
$$-4\Sigma q^n[\Sigma(2t+\tau)\cos(2t-\tau)x)],$$

$$\vartheta_1'\vartheta_3^3\vartheta_1(x)\vartheta_3(x)/\vartheta_0^2(x)\vartheta_2^2(x) = \tan x \sec x$$
$$+4\Sigma q^n[\Sigma(-1|\tau)(2t+\tau)\sin(2t-\tau)x],$$

$$\vartheta_1'\vartheta_2^3\vartheta_1(x)\vartheta_2(x)/\vartheta_0^2(x)\vartheta_3^2(x) = 4\Sigma q^{\beta/2}[(-1|\tau)(t+\tau)\sin(t-\tau)x],$$

$$\vartheta_1'\vartheta_2^3\vartheta_0(x)\vartheta_3(x)/\vartheta_1^2(x)\vartheta_2^2(x) = 4\csc^2 2x$$
$$-8\Sigma q^{2n}[\Sigma(-1)^\delta(d+\delta)\cos 2(d-\delta)x],$$

$$\vartheta_1'\vartheta_3^3\vartheta_0(x)\vartheta_2(x)/\vartheta_1^2(x)\vartheta_3^2(x) = \cot x \csc x$$
$$-4\Sigma q^n[(-1)^n\Sigma(2t+\tau)\cos(2t-\tau)x],$$

$$\vartheta_1'\vartheta_0^3\vartheta_0(x)\vartheta_1(x)/\vartheta_2^2(x)\vartheta_3^2(x) = \tan x \sec x$$
$$+4\Sigma q^n[(-1)^n\Sigma(-1|\tau)(2t+\tau)\sin(2t-\tau)x].$$

14. The theta equivalents of the twelve elliptic functions of Glaisher are found from $\phi_{abc}(x, 0)$:

$$\vartheta_2\vartheta_3\vartheta_1(x)/\vartheta_0(x) = 4\Sigma q^{m/2}[\Sigma \sin tx],$$
$$\vartheta_0\vartheta_2\vartheta_2(x)/\vartheta_0(x) = 4\Sigma q^{m/2}[\Sigma(-1|\tau)\cos tx].$$
$$\vartheta_0\vartheta_3\vartheta_3(x)/\vartheta_0(x) = 1+4\Sigma q^n[\Sigma(-1|\tau)\cos 2tx];$$

$$\vartheta_2\vartheta_3\vartheta_0(x)/\vartheta_1(x) = \csc x + 4\Sigma q^n[\Sigma \sin \tau x],$$
$$\vartheta_0\vartheta_3\vartheta_2(x)/\vartheta_1(x) = \cot x + 4\Sigma q^{2n}[\Sigma(-1)^\delta \sin 2dx],$$
$$\vartheta_0\vartheta_2\vartheta_3(x)/\vartheta_1(x) = \csc x + 4\Sigma q^n[(-1)^n\Sigma \sin \tau x];$$

$$\vartheta_0\vartheta_2\vartheta_0(x)/\vartheta_2(x) = \sec x + 4\Sigma q^n[(-1)^n\Sigma(-1|\tau)\cos \tau x],$$
$$\vartheta_0\vartheta_3\vartheta_1(x)/\vartheta_2(x) = \tan x - 4\Sigma q^{2n}[\Sigma(-1)^{d+\delta}\sin 2dx],$$
$$\vartheta_2\vartheta_3\vartheta_3(x)/\vartheta_2(x) = \sec x + 4\Sigma q^n[\Sigma(-1|\tau)\cos \tau x];$$

$$\vartheta_0\vartheta_3\vartheta_0(x)/\vartheta_3(x) = 1+4\Sigma q^n[(-1)^n\Sigma(-1|\tau)\cos 2tx],$$
$$\vartheta_0\vartheta_2\vartheta_1(x)/\vartheta_3(x) = 4\Sigma q^{m/2}[(-1|m)\Sigma \sin tx],$$
$$\vartheta_2\vartheta_3\vartheta_2(x)/\vartheta_3(x) = 4\Sigma q^{m/2}[\Sigma(-1|t)\cos tx].$$

15. From § 14 we get by the transformation of the second order

$$\vartheta_0^2\vartheta_0(x)\vartheta_1(x)/\vartheta_2(x)\vartheta_3(x)=\tan x-4\Sigma q^n\left[\Sigma(-1)^{d+\delta}\sin 2dx\right],$$
$$\vartheta_3^2\vartheta_0(x)\vartheta_2(x)/\vartheta_1(x)\vartheta_3(x)=\cot x+4\Sigma q^n\left[(-1)^n\Sigma(-1)^{\delta}\sin 2dx\right],$$
$$\vartheta_2^2\vartheta_0(x)\vartheta_3(x)/\vartheta_1(x)\vartheta_2(x)=2\csc 2x+8\Sigma q^{2n}\left[\Sigma\sin 2\tau x\right],$$
$$\vartheta_2^2\vartheta_1(x)\vartheta_2(x)/\vartheta_0(x)\vartheta_3(x)=8\Sigma q^m\left[\Sigma\sin 2\tau x\right],$$
$$\vartheta_3^2\vartheta_1(x)\vartheta_3(x)/\vartheta_0(x)\vartheta_2(x)=\tan x-4\Sigma q^n\left[(-1)^n\Sigma(-1)^{d+\delta}\sin 2dx\right],$$
$$\vartheta_0^2\vartheta^2(x)\vartheta_3(x)/\vartheta_0(x)\vartheta_1(x)=\cot x+4\Sigma q^n\left[\Sigma(-1)^{\delta}\sin 2dx\right].$$

16. The quadratic relations between the thetas applied to the derivatives of the six in § 15 give the squares of the functions in § 14. In these,

$$\mu_1(n)=\zeta_1(n)+\zeta_1'(n),\quad \nu_1(n)=\zeta_1'(n)-\zeta_1''(n),$$

where $\qquad \zeta_1(n),\quad \zeta_1'(n),\quad \zeta_1''(n)$

are respectively the sums of the first powers of all, of the odd, of the even, divisors of n.

$$\vartheta_2^2\vartheta_3^2\vartheta_0^2(x)/\vartheta_1^2(x)=4\Sigma q^n\mu_1(n)+\csc^2 x-8\Sigma q^{2n}\left[\Sigma d\cos 2dx\right],$$
$$\vartheta_0^2\vartheta_2^2\vartheta_0^2(x)/\vartheta_2^2(x)=4\Sigma q^n(-1)^n\mu_1(n)$$
$$\qquad\qquad+\sec^2 x-8\Sigma q^{2n}\left[\Sigma(-1)^d d\cos 2dx\right],$$
$$\vartheta_0^2\vartheta_3^2\vartheta_0^2(x)/\vartheta_3^2(x)=1+8\Sigma q^{2n}\nu_1(n)+8\Sigma q^n\left[(-1)^n\Sigma t\cos 2tx\right];$$

$$\vartheta_2^2\vartheta_3^2\vartheta_1^2(x)/\vartheta_0^2(x)=4\Sigma q^n\mu_1(n)-8\Sigma q^n\left[\Sigma t\cos 2tx\right],$$
$$\vartheta_0^2\vartheta_3^2\vartheta_1^2(x)/\vartheta_2^2(x)=-8\Sigma q^{2n}\nu_1(n)$$
$$\qquad\qquad+\tan^2 x-8\Sigma q^{2n}\left[\Sigma(-1)^d d\cos 2dx\right],$$
$$\vartheta_0^2\vartheta_2^2\vartheta_1^2(x)/\vartheta_3^2(x)=-4\Sigma q^n(-1)^n\mu_1(n)+8\Sigma q^n\left[(-1)^n\Sigma t\cos 2tx\right];$$

$$\vartheta_0^2\vartheta_2^2\vartheta_2^2(x)/\vartheta_0^2(x)=-4\Sigma q^n(-1)^n\mu_1(n)+8\Sigma q^n\left[\Sigma t\cos 2tx\right],$$
$$\vartheta_0^2\vartheta_3^2\vartheta_2^2(x)/\vartheta_1^2(x)=-8\Sigma q^{2n}\nu_1(n)+\cot^2 x-8\Sigma q^{2n}\left[\Sigma d\cos 2dx\right],$$
$$\vartheta_2^2\vartheta_3^2\vartheta_2^2(x)/\vartheta_3^2(x)=4\Sigma q^n\mu_1(n)-8\Sigma q^n\left[(-1)^n\Sigma t\cos 2tx\right];$$

$$\vartheta_0^2\vartheta_3^2\vartheta_3^2(x)/\vartheta_0^2(x)=1+8\Sigma q^{2n}\nu_1(n)+8\Sigma q^n\left[\Sigma t\cos 2tx\right],$$
$$\vartheta_0^2\vartheta_2^2\vartheta_3^2(x)/\vartheta_1^2(x)=4\Sigma q^n(-1)^n\mu_1(n)+\csc^2 x-8\Sigma q^{2n}\left[\Sigma d\cos 2dx\right],$$
$$\vartheta_2^2\vartheta_3^2\vartheta_3^2(x)/\vartheta_2^2(x)=4\Sigma q^n\mu_1(n)+\sec^2 x-8\Sigma q^{2n}\left[\Sigma(-1)^d d\cos 2dx\right].$$

17. The derivatives of the first four in § 16 give

$$\vartheta_1'^2\vartheta_1(x)\vartheta_2(x)\vartheta_3(x)/\vartheta_0^3(x) = 8\Sigma q^n[\Sigma t^2 \sin 2tx],$$

$$\vartheta_1'^2\vartheta_0(x)\vartheta_2(x)\vartheta_3(x)/\vartheta_1^3(x) = \csc^2 x \cot x - 8\Sigma q^{2n}[\Sigma d^2 \sin 2dx],$$

$$\vartheta_1'^2\vartheta_0(x)\vartheta_1(x)\vartheta_3(x)/\vartheta_2^3(x) = \sec^2 x \tan x$$
$$+ 8\Sigma q^{2n}[\Sigma(-1)^d d^2 \sin 2dx],$$

$$\vartheta_1'^2\vartheta_0(x)\vartheta_1(x)\vartheta_2(x)/\vartheta_3^3(x) = -8\Sigma q^n[(-1)^n \Sigma t^2 \sin 2tx].$$

18. The transformation of the second order applied to § 16 gives

$$\vartheta_0^4\vartheta_0^2(x)\vartheta_1^2(x)/\vartheta_2^2(x)\vartheta_3^2(x) = -8\Sigma q^n \nu_1(n) + \tan^2 x$$
$$-8\Sigma q^n[\Sigma(-1)^d d \cos 2dx],$$

$$\vartheta_3^4\vartheta_0^2(x)\vartheta_2^2(x)/\vartheta_1^2(x)\vartheta_3^2(x) = -8\Sigma q^n(-1)^n \nu_1(n)$$
$$+\cot^2 x - 8\Sigma q^n[(-1)^n \Sigma d \cos 2dx],$$

$$\vartheta_2^4\vartheta_0^2(x)\vartheta_3^2(x)/\vartheta_1^2(x)\vartheta_2^2(x) = 16\Sigma q^{2n}\mu_1(n)$$
$$+4\csc^2 2x - 32\Sigma q^{4n}[\Sigma d \cos 4dx],$$

$$\vartheta_2^4\vartheta_1^2(x)\vartheta_2^2(x)/\vartheta_0^2(x)\vartheta_3^2(x) = 16\Sigma q^{2n}\mu_1(n) - 32\Sigma q^{2n}[\Sigma t \cos 4tx],$$

$$\vartheta_3^4\vartheta_1^2(x)\vartheta_3^2(x)/\vartheta_0^2(x)\vartheta_2^2(x) = -8\Sigma q^n(-1)^n\nu_1(n) + \tan^2 x$$
$$-8\Sigma q^n[(-1)^n \Sigma(-1)^d d \cos 2dx],$$

$$\vartheta_0^4\vartheta_2^2(x)\vartheta_3^2(x)/\vartheta_0^2(x)\vartheta_1^2(x) = -8\Sigma q^n \nu_1(n)$$
$$+\cot^2 x - 8\Sigma q^n[\Sigma d \cos 2dx].$$

19. The derivatives of those in § 14 give

$$\vartheta_1'\vartheta_2\vartheta_1(x)\vartheta_2(x)/\vartheta_0^2(x) = 8\Sigma q^n[\Sigma(-1|\tau)\, t \sin 2tx],$$

$$\vartheta_1'\vartheta_3\vartheta_1(x)\vartheta_3(x)/\vartheta_0^2(x) = 4\Sigma q^{m/2}[\Sigma(-1|\tau)\, t \sin tx],$$

$$\vartheta_1'\vartheta_0\vartheta_2(x)\vartheta_3(x)/\vartheta_0^2(x) = 4\Sigma q^{m/2}[\Sigma t \cos tx];$$

$$\vartheta_1'\vartheta_3\vartheta_0(x)\vartheta_2(x)/\vartheta_1^2(x) = \csc x \cot x - 4\Sigma q^n[(-1)^n \Sigma\tau \cos \tau x],$$

$$\vartheta_1'\vartheta_2\vartheta_0(x)\vartheta_3(x)/\vartheta_1^2(x) = \csc^2 x - 8\Sigma q^{2n}[\Sigma(-1)^\delta d \cos 2dx],$$

$$\vartheta_1'\vartheta_0\vartheta_2(x)\vartheta_3(x)/\vartheta_1^2(x) = \csc x \cot x - 4\Sigma q^n[\Sigma\tau \cos \tau x];$$

$$\vartheta_1'\vartheta_0\vartheta_0(x)\vartheta_1(x)/\vartheta_2^2(x) = \sec x\tan x - 4\Sigma q^n[\Sigma(-1\,|\,\tau)\,\tau\sin\tau x],$$

$$\vartheta_1'\vartheta_2\vartheta_0(x)\vartheta_3(x)/\vartheta_2^2(x) = \sec^2 x - 8\Sigma q^{2n}[\Sigma(-1)^{d+\delta}d\cos 2dx],$$

$$\vartheta_1'\vartheta_3\vartheta_1(x)\vartheta_3(x)/\vartheta_2^2(x) = \sec x\tan x - 4\Sigma q^n[(-1)^n\Sigma(-1\,|\,\tau)\,\tau\sin\tau x];$$

$$\vartheta_1'\vartheta_0\vartheta_0(x)\vartheta_1(x)/\vartheta_3^2(x) = 4\Sigma q^{m/2}[\Sigma(-1\,|\,t)\,t\sin tx],$$

$$\vartheta_1'\vartheta_3\vartheta_0(x)\vartheta_2(x)/\vartheta_3^2(x) = 4\Sigma q^{m/2}[(-1\,|\,m)\,\Sigma t\cos tx],$$

$$\vartheta_1'\vartheta_2\vartheta_1(x)\vartheta_2(x)/\vartheta_3^2(x) = -8\Sigma q^n[(-1)^n\Sigma(-1\,|\,\tau)\,t\sin 2tx].$$

20. Applying to § 13 the transformation of the second order, changing q into $-q$ in some of the results, and, using in others, § 10, we get

$$\vartheta_1'\vartheta_2\vartheta_3\vartheta_1(x)/\vartheta_0^2(x) = \Sigma q^{\beta/4}[\Sigma(-1\,|\,\tau)(t+\tau)\sin\{\tfrac{1}{2}(t-\tau)x\}],$$

$$\vartheta_1'\vartheta_0\vartheta_2\vartheta_2(x)/\vartheta_0^2(x) = \Sigma q^{\beta/4}[\Sigma(t+\tau)\cos\{\tfrac{1}{2}(t-\tau)x\}],$$

$$\vartheta_1'\vartheta_0\vartheta_3\vartheta_3(x)/\vartheta_0^2(x) = 2\Sigma q^{\alpha/4}[\Sigma t\cos\{\tfrac{1}{2}(t-\tau)x\}];$$

$$\vartheta_1'\vartheta_2\vartheta_3\vartheta_0(x)/\vartheta_1^2(x) = \csc^2 x - 2\Sigma q^n[\Sigma(-1)^\delta(d+\delta)\cos(d-\delta)x],$$

$$\vartheta_1'\vartheta_0\vartheta_3\vartheta_2(x)/\vartheta_1^2(x) = \cot x\csc x - 4\Sigma q^{2n}[\Sigma(2t+\tau)\cos(2t-\tau)x],$$

$$\vartheta_1'\vartheta_0\vartheta_2\vartheta_3(x)/\vartheta_1^2(x) = \csc^2 x - 2\Sigma q^n[(-1)^n\Sigma(-1)^\delta(d+\delta)\cos(d-\delta)x];$$

$$\vartheta_1'\vartheta_0\vartheta_2\vartheta_0(x)/\vartheta_2^2(x) = \sec^2 x - 8\Sigma q^{4n}[\Sigma(-1)^{d+\delta}d\cos 2(d-\delta)x] - 4\Sigma q^m[(-1\,|\,m)\,\Sigma t\cos(t-\tau)x],$$

$$\vartheta_1'\vartheta_0\vartheta_3\vartheta_1(x)/\vartheta_2^2(x) = \tan x\sec x + 4\Sigma q^{2n}[(-1)^n\Sigma(-1\,|\,\tau)(2t+\tau)\sin(2t-\tau)x],$$

$$\vartheta_1'\vartheta_2\vartheta_3\vartheta_3(x)/\vartheta_2^2(x) = \sec^2 x - 8\Sigma q^{4n}[\Sigma(-1)^{d+\delta}d\cos 2(d-\delta)x] + 4\Sigma q^m[(-1\,|\,m)\,\Sigma t\cos(t-\tau)x];$$

$$\vartheta_1'\vartheta_0\vartheta_3\vartheta_0(x)/\vartheta_3^2(x) = 2\Sigma q^{\alpha/4}[(2\,|\,\alpha)\,\Sigma t\cos\{\tfrac{1}{2}(t-\tau)x\}],$$

$$\vartheta_1'\vartheta_0\vartheta_2\vartheta_1(x)/\vartheta_3^2(x) = -\Sigma q^{\beta/4}[(2\,|\,\beta)\,\Sigma(-1\,|\,\tau)(t+\tau)\sin\{\tfrac{1}{2}(t-\tau)x\}],$$

$$\vartheta_1'\vartheta_2\vartheta_3\vartheta_2(x)/\vartheta_3^2(x) = -\Sigma q^{\beta/4}[(2\,|\,\beta)\,\Sigma\,t+\tau)\cos\{\tfrac{1}{2}(t-\tau)x\}].$$

21. The four $\phi_{abc}(x, x)$ with $a=1$ give

$$\vartheta_1(x)\vartheta_2(x)\vartheta_3(x)/\vartheta_0(x) = 2\Sigma q^{m/2}[\Sigma \sin(t+\tau)x],$$
$$\vartheta_0(x)\vartheta_2(x)\vartheta_3(x)/\vartheta_1(x) = \cot x + 2\Sigma q^{2n}[\Sigma \sin 2(d+\delta)x],$$
$$\vartheta_0(x)\vartheta_1(x)\vartheta_3(x)/\vartheta_2(x) = \tan x - 2\Sigma q^{2n}[\Sigma(-1)^{d+\delta}\sin 2(d+\delta)x],$$
$$\vartheta_0(x)\vartheta_1(x)\vartheta_2(x)/\vartheta_3(x) = 2\Sigma q^{m/2}[(-1\,|\,m)\,\Sigma \sin(t+\tau)x].$$

22. Adding or subtracting pairs in § 21 and applying to the results the transformations of the first and second order we find

$$\vartheta_3\vartheta_1(x)\vartheta_3(x)/\vartheta_0(x) = 2\Sigma q^{\alpha/4}[\Sigma \sin\{\tfrac{1}{2}(t+\tau)x\}],$$
$$\vartheta_0\vartheta_2(x)\vartheta_3(x)/\vartheta_0(x) = 2\Sigma q^{\alpha/4}[\Sigma(-1\,|\,\tau)\cos\{\tfrac{1}{2}(t+\tau)x\}];$$

$$\vartheta_3\vartheta_0(x)\vartheta_2(x)/\vartheta_1(x) = \cot x + 2\Sigma q^{m}[\Sigma \sin(t+\tau)x]$$
$$+ 2\Sigma q^{4n}[\Sigma \sin 2(d+\delta)x],$$
$$\vartheta_0\vartheta_2(x)\vartheta_3(x)/\vartheta_1(x) = \cot x - 2\Sigma q^{m}[\Sigma \sin(t+\tau)x]$$
$$+ 2\Sigma q^{4n}[\Sigma \sin 2(d+\delta)x];$$
$$\vartheta_0\vartheta_0(x)\vartheta_1(x)/\vartheta_2(x) = \tan x - 2\Sigma q^{m}[(-1\,|\,m)\,\Sigma \sin(t+\tau)x]$$
$$- 2\Sigma q^{4n}[\Sigma(-1)^{d+\delta}\sin 2(d+\delta)x],$$
$$\vartheta_3\vartheta_1(x)\vartheta_3(x)/\vartheta_2(x) = \tan x + 2\Sigma q^{m}[(-1\,|\,m)\,\Sigma \sin(t+\tau)x]$$
$$- 2\Sigma q^{4n}[\Sigma(-1)^{d+\delta}\sin 2(d+\delta)x];$$

$$\vartheta_0\vartheta_0(x)\vartheta_1(x)/\vartheta_3(x) = 2\Sigma q^{\alpha/4}[(2\,|\,\alpha)\,\Sigma \sin\{\tfrac{1}{2}(t+\tau)x\}],$$
$$\vartheta_3\vartheta_0(x)\vartheta_2(x)/\vartheta_3(x) = 2\Sigma q^{\alpha/4}[(2\,|\,\alpha)\,\Sigma(-1\,|\,\tau)\cos\{\tfrac{1}{2}(t+\tau)x\}].$$

23. By changes of x into $x+\pi\omega/2$, where ω is the ratio of the periods of the theta functions, the four missing members of the list in § 22 can be found. But, as the necessary reductions belong more properly to the arithmetical transformation of Biehler's series, we shall omit them. The analytical forms of the lists in §§ 20, 21, 22 were otherwise obtained by Hermite or Biehler. The remaining $\phi_{abc}(x, x)$, also the $\phi_{abc}(2x, -x)$, give certain others of Biehler's more complicated expansions in reduced form.

A NOTE ON THE POLYNOMIAL FACTOR OCCURRING IN THE ABNORMAL CASE OF THE APPROXIMATION TO A FUNCTION BY MEANS OF A RATIONAL FRACTION.

By *R. Wilson*, University College, Swansea.

§ 1. *Introductory.*

LET $\dfrac{a_0+a_1z+\ldots+a_\mu z^\mu}{b_0+b_1z+\ldots+b_\nu z^\nu}\equiv\dfrac{P}{Q}$ be a rational algebraic fraction, the expansion of which agrees with that of the given function

$$f(z)\equiv c_0+c_1z+c_2z^2+\ldots+c_nz^n+\ldots$$

for the first $(\mu+\nu+1)$ terms. Then

$$f(z)-\frac{a_0+a_1z+\ldots+a_\mu z^\mu}{b_0+b_1z+\ldots+b_\nu z^\nu}=O(z^{\mu+\nu+1}).$$

Therefore

$$(b_0+b_1z+\ldots+b_\nu z^\nu)(c_0+c_1z+\ldots+c_{\mu+\nu}z^{\mu+\nu}+\ldots)$$
$$-(a_0+a_1z+\ldots+a_\mu z^\mu)=O(z^{\mu+\nu+1})\ldots\ldots(1).$$

Hence

$$\left.\begin{array}{ll} a_r=b_0c_r+\ldots+b_\nu c_{r-\nu} & (r=0,\ 1,\ 2,\ \ldots,\ \mu) \\ 0=b_0c_r+\ldots+b_\nu c_{r-\nu} & (r=\mu+1,\ \ldots,\ \mu+\nu),\quad c_{-m}=0 \end{array}\right\}\ldots(2).$$

This fraction is uniquely determinate unless the matrix

$$\begin{vmatrix} c_{\mu+1} & \cdots & c_{\mu-\nu+1} \\ \\ c_{\mu+\nu} & \cdots & c_\mu \end{vmatrix}=0.$$

In this case some of the b's may be taken arbitrarily, *e.g.* any one b may be taken to be zero.

Padé* calls that solution, for which the longest sequence of b_0, b_1, b_2, ... is zero, the Principal Solution. (Hereafter referred to as the P.S.).

* *Annales de l'Ecole Normale Supérieure*, ser. 3, vol. ix. (1892), supplement pp. 1–93.

It will be shown that if $\frac{z^\omega U}{z^\omega V}$ is the fraction corresponding to the P.S., then $P \equiv KU$, $Q \equiv KV$, where K is some polynomial in z.

In this paper we shall discuss the possible forms of K, and the nature of the solutions at any point in the abnormal field.

§ 2. *The fraction arising from the P.S. is irreducible (with the exception of the factor z^ω).*

For otherwise we obtain a fraction with the given order of approximation for which the sequence of zeros $b_0, b_1, b_2, \ldots$ exceeds that of the P.S., and this fraction must be a solution, since it satisfies a relation of the type (1). Hence the original solution cannot be the P.S.

§ 3. *The determination of the P.S.*

Suppose the P.S. given. Then

$$\left.\begin{array}{l} a_r = b_\omega c_{r-\omega} + \ldots + b_\nu c_{r-\nu} \quad (r = \omega,\ \omega+1,\ \ldots,\ \mu) \\ 0 = b_\omega c_{r-\omega} + \ldots + b_\nu c_{r-\nu} \quad (r = \mu+1,\ \ldots,\ \mu+\nu),\ c_{-m} = 0 \end{array}\right\} \ldots (3).$$

Hence the matrix

$$\begin{vmatrix} c_{\mu+1-\omega} & \ldots & c_{\mu-\nu+1} \\ c_{\mu+\nu-\omega} & \ldots & c_\mu \end{vmatrix} \text{ in } \begin{vmatrix} c_{\mu+1} & \ldots & c_{\mu-\nu+1} \\ c_{\mu+\nu} & \ldots & c_\mu \end{vmatrix}$$

is zero. The matrix

$$\begin{vmatrix} c_{\mu-\omega} & \ldots & c_{\mu-\nu+1} \\ c_{\mu+\nu-1-\omega} & \ldots & c_\mu \end{vmatrix}$$

cannot be zero, for this would imply the existence of a solution with a higher order of zeros than the P.S.

Hence the rank of the matrix

$$\begin{vmatrix} c_{\mu+1} & \ldots & c_{\mu-\nu+1} \\ c_{\mu+\nu} & \ldots & c_\mu \end{vmatrix}$$

is $(\nu-\omega)$, and the determinantal expressions for $b_0, b_1, b_2, \ldots, b_{\omega-1}$ are, as expected, zero.

The P.S. can be determined by taking successive matrices of the form

$$\begin{vmatrix} c_{\mu-\nu+1+k} & \cdots & c_{\mu-\nu+1} \\ \\ c_{\mu+k} & \cdots & c_{\mu} \end{vmatrix}$$

for $h=0, 1, 2, \ldots$ in succession until a matrix is obtained which does not vanish. This matrix determines ω $(=\nu-h)$, and from this the P.S. is determinate by (3).

§ 4. *The other solutions.*

Consider a solution, not the P.S.,

$$\frac{a_0'+a_1'z+\ldots+a_\mu'z^\mu}{b_0'+b_1'z+\ldots+b_\nu'z^\nu}\equiv\frac{P}{Q}$$

(of which some of the a''s and b''s may be zero), approximating to the function to the order $(\mu+\nu+k+1)$. (In the normal case $k=0$).

Then, as in § 1,

$$\left.\begin{array}{l} a_r'=b_0'c_r+\ldots+b_\nu'c_{r-\nu}\ \ (r=0,\ \ldots,\ \mu) \\ \ 0=b_0'c_r+\ldots+b_\nu'c_{r-\nu}\ \ (r=\mu+1,\ \ldots,\ \mu+\nu+k),\ c_{-m}=0 \end{array}\right\}\ldots(4),$$

and therefore the specialised condition

$$\begin{vmatrix} c_{\mu+1} & \cdots & c_{\mu-\nu+1} \\ \cdots & \cdots & \cdots \\ c_{\mu+\nu+k} & \cdots & c_{\mu+k} \end{vmatrix}=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

must hold.

Let $0, 0, \ldots, 0, b_\omega, \ldots, b_\nu$ be the P.S. obtained as in § 3.

Thus $0, 0, \ldots, 0, b_\omega, \ldots, b_\nu$ is a solution of equations (4) for $r=\mu+1, \ldots, \mu+\nu$.

Hence $0, 0, \ldots, b_\omega, b_{\omega+1}, \ldots, b_\nu, 0$ is a solution of equations (4) for $r=\mu+2, \ldots, \mu+\nu+1$,

and $0, 0, \ldots, b_\omega, b_{\omega+1}, \ldots, b_\nu, 0, 0$ is a solution of equations (4) for $r=\mu+3, \ldots, \mu+\nu+2$,

....

Finally $0, \ldots, b_\omega, \ldots, b_\nu, 0, \ldots, 0, 0$ is a solution of equations (4) for $r=\mu+k+1, \ldots, \mu+\nu+k$.

These solutions are independent, and are the 'fundamental set' for equations (1). The most general solution of equations (4) is therefore

$$b_r' = b_{k+r}d_{\omega-k} + \ldots + b_r d_\omega, \text{ where } b_s = 0,\ s < \omega \text{ or } s > \nu \ldots (6),$$

and the d's are *arbitrary*.

This gives

$$(b_0' + b_1'z + \ldots + b_\nu' z^\nu) = (b_\omega + \ldots + b_\nu z^{\nu-\omega})(d_{\omega-k}z^{\omega-k} + \ldots + d_\omega z^\omega).$$

So $\quad a_r' = (b_0'c_r + \ldots + b_\nu' c_{r-\nu}) = b'_{\omega-k}c_{r-\omega+k} + \ldots + b_\nu' c_{r-\nu}$

$$= (b_\omega d_{\omega-k} + \ldots + b_{\omega-k}d_\omega)\, c_{r-\omega+k} + \ldots + (b_\nu d_{\omega-k} + \ldots + b_{\nu-k}d_\omega)$$

$$= a_{r+k}d_{\omega-k} + \ldots + a_r d_\omega.$$

Hence

$$(a_0' + a_1'z + \ldots + a_\mu' z^\mu) = (a_\omega + \ldots + a_\mu z^{\mu-\omega})(d_{\omega-k}z^{\omega-k} + \ldots + d_\omega z^\omega).$$

Taking into account the matrix condition of § 3 giving the P.S., condition (5) can be written

$$\begin{vmatrix} c_{\mu-\omega} & \ldots & c_{\mu-\nu} \\ \ldots & \ldots & \ldots \\ c_{\mu+\nu+k-\omega} & \ldots & c_{\mu+k} \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (7).$$

Therefore, if this condition holds,

(A) The P.S. has the form $\dfrac{z^\omega U}{z^\omega V}$, where $\dfrac{U}{V}$ is irreducible.

(B) Any other solution is $\dfrac{KU}{KV}$, where $K \equiv d_{\omega-k}z^{\omega-k} + \ldots + d_\omega z^\omega$, and the d's are arbitrary. Call K a polynomial of 'order' k.

For a solution under condition (7), b_ω', ..., $b'_{\omega-k+1}$ and the corresponding a''s are all zero.

We can, instead of taking the kd's arbitrarily, take the first $(\omega - k)$ b's to be zero, and k of the remaining b's to be arbitrary.

The k d's are then determined from equations (6).

It is to be noted that K is of order k if $\omega - k \geq 0$, and of order ω if $\omega - k \leq 0$, *i.e.* the order of K is the smaller of k and ω.

§ 5. *The forms of the polynomial K at points in the abnormal field.*

Suppose the condition

$$\begin{vmatrix} c_{\mu+1-\omega} & \cdots & c_{\mu-\nu+1} \\ \cdots & \cdots & \cdots \\ c_{\mu+\nu-\omega} & \cdots & c_{\mu} \end{vmatrix} = 0 \quad \text{...............(8)},$$

given where the rank of the matrix is $(\nu-\omega)$. Then every fraction (μ', ν'), where $\mu \geq \mu' \geq \mu-\omega$; $\nu \geq \nu' \geq \nu-\omega$, has a P.S. of degrees $(\mu-\omega, \nu-\omega)$ by § 3, and this must be the normal solution at the point $(\mu-\omega, \nu-\omega)$. We shall represent the fraction, which before reduction has degrees μ', ν' for numerator and denominator, by the point with coordinates (μ', ν'). Hence all fractions whose index points are contained in the square $(\mu-\omega, \nu-\omega)$, $(\mu, \nu-\omega)$, (μ, ν), $(\mu-\omega, \nu)$ have the same P.S., and the common fraction to which the fractions at all such points reduce is the normal solution at the point $(\mu-\omega, \nu-\omega)$. The corners of the square indexed above are the points A, B, C, D in the same order in the diagram.

For the point (μ, ν), $k=0$, and $K \equiv d_\omega z^\omega$; every solution at this point is identical with the P.S.

At the point $(\mu-\omega, \nu-\omega)$ the solution is normal and $K \equiv d_0$.

Consider the solution at the point P, $(\mu-l, \nu-l)$ of the diagonal AC. For this

$$\mu' = \mu - l,\ \nu' = \nu - l \text{ and } \omega' = (\mu - l) - (\mu - \omega) = \omega - l.$$

Condition (8) becomes

$$\begin{vmatrix} c_{\mu'+1-\omega'} & \cdots & c_{\mu'-\nu'+1} \\ \cdots & \cdots & \cdots \\ c_{\mu'+\nu'+l-\omega'} & \cdots & c_{\mu'+l} \end{vmatrix} = 0.$$

Whence by (7) $k=l$, and $K \equiv z_{\omega-2l}(d_{\omega-2l}+\ldots+d_{\omega-l}z^l)$, when $\omega-2l \geq 0$. If $\omega-2l<0$, *i.e.* $\omega'<l$, K is of order $\omega'=\omega-l$.

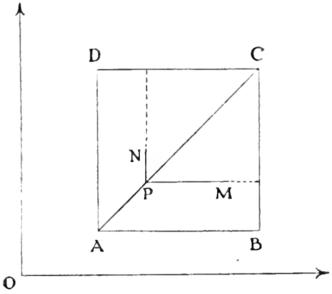
Considering only fractions whose index points are on the diagonal AC, we have the following scheme.

Index point.

$(\mu-\omega,\ \nu-\omega)$; $(\mu-\omega+1,\ \nu-\omega+1)$; ..., $(\mu-1,\ \nu-1)$; $(\mu,\ \nu)$.

Polynomial K. d_0; (d_0+d_1z); ..., $z^{\omega-2}(d_{\omega-2}+d_{\omega-1}z)$; $d_\omega z^\omega$,

the d's being arbitrary in every case, and therefore in no way related. Thus the polynomial K has the same degree of



arbitrariness at the end points of the diagonal AC, and at all pairs of points equidistant from the two ends, and this degree increases by unity at each step taken along the diagonal, beginning from either end.

If ω is odd, K attains its highest order, $\{\frac{1}{2}(\omega-1)\}$, at the centre point.

If ω is even, K attains its highest order, $(\frac{1}{2}\omega-1)$, at the two most central points.

Consider now the solution at any point M, $(\mu-l+m,\ \nu-l)$ of the triangle ABC. For this $\omega'=\omega-l$, since the denominator is reduced in degree from $(\nu-l)$ to $(\nu-\omega)$.

Substituting

$$\mu'=\mu-l+m,\ \nu'=\nu-l,\ \omega'=\omega-l,$$

and applying conditions (8), we get

$$\begin{vmatrix} c_{\mu'+m+1-\omega'} & \cdots & c_{\mu'-\nu'+m+1} \\ \cdots & \cdots & \cdots \\ c_{\mu'+\nu'+(l-m)-\omega'} & \cdots & c_{\mu'+(l-m)} \end{vmatrix} = 0.$$

Condition (7) gives $k=(l-m)$. Whence K is of order $k=(l-m)$ or $\omega'=(\omega-l)$, whichever is the smaller.

For the solution at any point N, $(\mu-l,\ \nu-l+n)$ of the triangle ADC, $\omega'=\omega-l$, since the numerator is reduced from the degree $(\mu-l)$ to $(\mu-\omega)$. Substituting

$$\mu'=\mu-l,\quad \nu'=\nu-l+n,\quad \omega'=\omega-l,$$

condition (8) gives

$$\begin{vmatrix} c_{\mu'+1-\omega'} & \cdots & c_{\mu'-\nu'+n+1} \\ \cdots & \cdots & \cdots \\ c_{\mu'+\nu'+(l-n)} & \cdots & c_{\mu'+l} \end{vmatrix} = 0.$$

From condition (7), $k=(l-n)$. Whence the order of K is the smaller of $k=(l-n)$ and $\omega'=(\omega-l)$. The order of K and the form of the solution are thus given for every point of the field of abnormality.

§ 6. *Some examples.*

In determining the areas of abnormality associated with the rational fractions approximating to a function in the scheme of § 5 (known as the Padé table), it is necessary, in general, to evaluate the fraction corresponding to every index point. Conditions (7) and (8) are, however, both necessary and sufficient conditions, and hence, if every Hankel determinant formed from the successive coefficients in the Taylor expansion of the function is non-zero, the fractions are normal at every point of the table.

Example 1. We shall evaluate the determinant

$$\begin{vmatrix} c_{\mu+1} & \cdots & c_{\mu-\nu+1} \\ \cdots & \cdots & \cdots \\ c_{\mu+\nu+1} & \cdots & c_{\mu+1} \end{vmatrix}$$

for each of the three functions e^z; $(1-z)^m$ (m non-integral); and $-\dfrac{1}{z}\log_e(1-z)$.

The values of these determinants are respectively $(\mu \geq \nu)$ to within the factor $(-1)^{\frac{1}{2}\{\nu(\nu+1)\}}$

$$-1!\,2!\ldots\nu!/(\mu-\nu+1)!\ldots(\mu+1)!\,(\mu-\nu+2)\,(\mu-\nu+3)^2\ldots$$
$$\ldots(\mu+1)^\nu(\mu+2)^\nu\ldots(\mu+\nu)^2\,(\mu+\nu+1);$$

$${}^mC_{\mu-\nu+1}\,{}^mC_{\mu-\nu+2}\ldots{}^mC_{\mu+1}\,(1!\ldots\nu!)\,(1+m)^\nu\,(2+m)^{\nu-1}$$

$$\ldots(\nu+m)\,/\,(\mu-\nu+2)\,(\mu-\nu+3)^2\ldots(\mu+\nu)^2\,(\mu+\nu+1)\,;$$

$$(1!\,2!\ldots\nu!)^2\,/\,(\mu-\nu+1)(\mu-\nu+2)^2\ldots(\mu+1)^{\nu+1}\ldots(\mu+\nu)^2(\mu+\nu+1);$$

where ${}^mC_n$ is the binomial coefficient.

The determinants in all cases are clearly non-zero for finite positive integral values of μ and ν such that $\mu \geq \nu$. The values for $\mu < \nu$ are easy to obtain but are very cumbersome. They are also non-zero for the range of integral values of μ and ν considered. Hence all the three functions e^z, $(1-z)^m$ (m non-integral), and $-\frac{1}{z}\log_e(1-z)$ possess tables of fractions normal at every point.

Example 2. The algebraic rational fraction provides a different type of example. This function can be written

$$\phi(z) + \Sigma \frac{A_k}{(1-a_k z)^m} \equiv c_0 + c_1 z + c_2 z^2 + \ldots + c_r z^r + \ldots,$$

where $\phi(z)$ is a polynomial and the number of terms in Σ is finite.

The development clearly gives a recurring series, and hence the relation $c_{n+r} + p_1 c_{n+r-1} + \ldots + p_r c_n = 0$ holds for every integral $n \geq$ some particular integer m.

Therefore
$$\begin{vmatrix} c_{m+r} & \ldots & c_m \\ c_{m+r+1} & \ldots & c_{m+1} \\ \ldots & \ldots & \ldots \end{vmatrix} = 0,$$

where the matrix is infinite in extent. The rank cannot be less than $(r+1)$, for this would imply the existence of a different recurrence relation from that given.

Comparing with condition (8), we get

$$\mu - \nu + 1 = m, \quad \mu + 1 - \omega = m + r,$$

and μ is infinite. Hence μ and ω are infinite. Also

$$\mu - \omega = m + r - 1, \quad \nu - \omega = r.$$

The square of abnormality is therefore infinite in extent and has the one finite vertex $(m + r - 1, r)$.

ON CURVES FOR WHICH ALL LINES THROUGH A FIXED POINT ARE COMPONENTS OF FIRST POLARS.

By *C. H. Sisam.*

It is the purpose of this paper to determine the form of the equations of all non-composite algebraic plane curves having the property that every right line of the pencil through a fixed point is a component of the first polar of some point with respect to the given curve.

We take the fixed point as the vertex (0, 1, 0) of the triangle of reference, and denote by V_k a binary form of degree k in x and t.

It will be shown that:

(i) If the given curve of order n $(n > 1)$ has an $(n-1)$-fold point at (0, 1, 0), the conditions of the problem are always satisfied. In this case the equations of the curve are

$$V_{n-1}y + V_n = 0.$$

(ii) If (0, 1, 0) does not lie on the curve and $n > 1$, the condition is that the equation of the curve can be reduced by a suitable transformation of coordinates to the form

$$V_0 y^n + V_n = 0.$$

(iii) If (0, 1, 0) is an $(n-m)$-fold point on the surface $(n > m > 1)$, then we must have $n = m(mq+1)$, and the equation of the curve must be reducible to the form

$$(V_m{}^q y + V_{mq+1})^m + k V_m{}^{mq+1} = 0.$$

To one of the above three types the equation of any curve that satisfies the conditions of the problem can be reduced.

To prove these statements we write the equation of the curve in the form

$$f(x, y, t) \equiv V_i y^m + m V_{i+1} y^{m-1} + \dots$$
$$+ {}_mC_j V_{i+j} y^{m-j} + \dots + V_{i+m} = 0 \dots (1),$$

wherein the ${}_mC_j$ are binomial coefficients. We use the notation $\dfrac{\partial V_k}{\partial x} = V_{kx}$ and $\dfrac{\partial V_k}{\partial t} = V_{kt}$ and write the equation of the first polar of the point (x', y', t') as

$$\begin{aligned}(x' V_{ix} + t' V_{it})\, y^m + m\,(x' V_{(i+1)x} + y' V_i + t' V_{(i+1)t})\, y^{m-1} + \dots \\ + {}_mC_j\,(x' V_{(i+j)x} + y' j V_{i+j-1} + t' V_{(i+j)t})\, y^{m-j} + \dots \\ + x' V_{(i+m)x} + y' m V_{i+m-1} + t' V_{(i+m)t} = 0 \quad \dots\dots(2).\end{aligned}$$

The condition that every line through (0, 1, 0) is a component of a first polar curve (2) is that, for every value of the ratio $x : t$, there exists a set of values of the ratios $x' : y' : t'$ that makes the coefficients of every power of y in (2) vanish; that is, that the equations

$$x' V_{(i+j)x} + j y' V_{i+j-1} + t' V_{(i+j)t} = 0, \quad j = 0, 1, \dots, m$$

are consistent (x', y', t' not all zero) for every value of the ratio $x : t$. The necessary and sufficient condition for this is that every third order determinant in the matrix

$$\left\|\begin{matrix} V_{ix} & V_{(i+1)x} & V_{(i+2)x} \dots V_{(i+j)x} \dots V_{(i+m)x} \\ 0 & V_i & 2V_{i+1} \dots j V_{i+j-1} \dots m V_{i+m-1} \\ V_{it} & V_{(i+1)t} & V_{(i+2)t} \dots V_{(i+j)t} \dots V_{(i+m)t} \end{matrix}\right\| \quad \dots\dots(3)$$

shall vanish identically.

(i) If $m = 1$ the condition on (3) is always satisfied; that is, *every line through an i-fold point* ($i > 0$) *on a curve of order $i + 1$ is a component of a first polar with respect to the curve.*

The locus of a point whose first polar contains such a rectilinear component is a rational curve of order (in general) $2i - 1$, which has an i-fold point at (0, 1, 0). It is defined parametrically by the equations

$$x' = V_i V_{it}, \quad y' = V_{ix} V_{(i+1)t} - V_{it} V_{(i+1)x}, \quad t' = -V_i V_{ix}.$$

(ii) If $i = 0$, *i.e.* if (0, 1, 0) does not lie on the curve, we choose its polar line as $y = 0$. Then $V_1 = 0$. Since, further, $V_0 \neq 0$, the condition on (3) is that every second order determinant in the matrix

$$\left\|\begin{matrix} V_{2x} & V_{3x} \dots V_{mx} \\ V_{2t} & V_{3t} \dots V_{mt} \end{matrix}\right\| \quad \dots\dots\dots\dots\dots\dots(4)$$

shall vanish identically.

Since $f(x, y, t)=0$ is not composite, it follows that its equation reduces to

$$V_0 y^m + V_m = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

For, on account of (4), we have

$$V_2 = k_2 V_m^{\frac{2}{m}}, \quad V_3 = k_3 V_m^{\frac{3}{m}}, \ \ldots, \ V_{m-1} = k_{m-1} V_m^{\frac{m-1}{m}},$$

where k_2, k_3, etc., are constants of integration.

Let l be the lowest power of V_m such that $V_m^{\frac{l}{m}}$ is rational. Then every k_j is zero except those for which j is a multiple of l. If we let $V_m^{\frac{l}{m}} = \sigma$, the equation of the curve is

$$V_0 y^m + k_l \sigma y^{m-l} + k_{2l} \sigma^2 y^{m-2l} + \ldots + k_{\frac{m}{l}} \sigma^{\frac{m}{l}} = 0.$$

This equation is homogeneous in y^l and σ, and thus factors rationally into

$$V_0 (y^l - r_1 \sigma)(y^l - r_2 \sigma) \ldots \quad = 0.$$

Hence, since the given curve is not composite, $l = m$, and the form of the equation is given by (5).

(iii) In considering the general case, $m > 1$ and $i > 0$, we first reduce the matrix (3) to the form

$$\left\| \begin{matrix} dV_i & dV_{i+1} & dV_{i+2} \ldots\ldots\ldots dV_{i+j} \ldots\ldots\ldots dV_{i+m} \\ 0 & V_i & 2V_{i+1} \ldots\ldots\ldots jV_{i+j-1} \ldots\ldots\ldots mV_{i+m-1} \\ iV_i & (i+1)V_{i+1} & (i+2)V_{i+2} \ldots (i+j)V_{i+j} \ldots (i+m)V_{i+m} \end{matrix} \right\| \ldots (6)$$

by using Euler's theorem, $(i+j)V_{i+j} = xV_{(i+j)x} + tV_{(i+j)t}$, and then putting $t=1$.

It will first be shown that the equation of a curve for which the polynomials V_{i+j} make all the third order determinants in (6) vanish can be written in the form

$$V_i \left(y + \frac{V_{i+1}}{V_i}\right)^m + k_2 V_i V_i^{\frac{2}{i}} \left(y + \frac{V_{i+1}}{V_i}\right)^{m-1} + \ldots$$

$$+ k_j V_i V_i^{\frac{j}{i}} \left(y + \frac{V_{i+1}}{V_i}\right)^{m-j} + \ldots + k_m V_i V_i^{\frac{m}{i}} = 0.$$

The proof is by induction. The theorem is true for $m=2$, since, by integration from (6),

$$V_{i+2} = \frac{V^2_{i+1}}{V_i} + k_2 V_i V_i^{\frac{2}{i}}.$$

Suppose the theorem true for all values of m less than the given value. Then, since the coefficients of

$$\frac{1}{m}\frac{\partial b}{\partial y} = V_i y^{m-1} + (m-1) V_{i+1} y^{m-2} + \ldots + {}_{m-1}C_j V_{i+j} y^{m-j-1} + \ldots + V_{i+m-1}$$

satisfy (6), we have

$$\frac{\partial b}{\partial y} = m V_i \left(y + \frac{V_{i+1}}{V_i}\right)^{m-1} + (m-2) k_2 V_i V_i^{\frac{2}{i}} \left(y + \frac{V_{i+1}}{V_i}\right)^{m-3} + \ldots$$
$$+ (m-j) k_j V_i V_i^{\frac{j}{i}} \left(y + \frac{V_{i+1}}{V_i}\right)^{m-j-1} + \ldots + k_{m-1} V_i V_i^{\frac{m-1}{i}}.$$

By integration we find

$$f(x, y, t) = V_i \left(y + \frac{V_{i+1}}{V_i}\right)^m + k_2 V_i V_i^{\frac{2}{i}} \left(y + \frac{V_{i+1}}{V_i}\right)^{m-2} + \ldots$$
$$+ k_j V_i V_i^{\frac{j}{i}} \left(y + \frac{V_{i+1}}{V_i}\right)^{m-j} + \ldots + k_{m-1} V_i V_i^{\frac{m-1}{i}} \left(y + \frac{V_{i+1}}{V_i}\right) + O_{m+i} = 0,$$

where the value of O_{m+i} is found from (6) to be $k_m V_i V_i^{\frac{m}{i}}$.

Let l be the lowest power V_i such that $V_i^{\frac{l}{i}}$ is rational. Since f is rational, all values of k_j, such that j is not a multiple of l, are zero. We may now write the equation of f in the form

$$\frac{1}{V_i^{m-1}} \left[(V_i y + V_{i+1})^m + k_l V_i^{\frac{l}{i}} (V_i y + V_{i+1})^{m-l} + \ldots + k_m V_i^{m+\frac{m}{i}}\right] = 0.$$

Since the curve $f = 0$ is not composite, we have $l = m$. The equation now reduces to

$$V_i \left(y + \frac{V_{i+1}}{V_i}\right)^m + k_m V_i V_i^{\frac{m}{i}} = 0.$$

Since $f(x, y, t)$ is integral and not composite, $\frac{V_{i+1}^m}{V_i^{m-1}}$ is integral and contains none of the factors of V_i. Hence, V_i is a perfect m^{th} power. For, let

$$V_i = a_1^{\alpha_1} a_2^{\alpha_2} \ldots a_\lambda^{\alpha_\lambda}$$

and

$$V_{i+1} = a_1^{\alpha_1'} a_2^{\alpha_2'} \ldots a_{\lambda'}^{\alpha'_{\lambda'}},$$

where $\lambda' \geq \lambda$ and $a_j = b_j x + c_j t$. Then

$$\frac{V_{i+1}^{\ m}}{V_i^{m-1}} = \frac{a_1^{\ m\alpha_1'} a_2^{\ m\alpha_2'} \ldots a_{\lambda'}^{\ m\alpha'_{\lambda'}}}{a_1^{\ (m-1)\alpha_1} a_2^{\ (m-1)\alpha_2} \ldots a_\lambda^{\ (m-1)\alpha_\lambda}}.$$

Since this expression is integral and free from $a_1, a_2, \ldots, a_\lambda$,

$$m\alpha_1' = (m-1)\,\alpha_1, \quad m\alpha_2' = (m-1)\,\alpha_2, \ \ldots, \ m\alpha_\lambda' = (m-1)\,\alpha_\lambda.$$

But m and $m-1$ are relatively prime, so that

$$\alpha_1 = m\alpha_1'', \quad \alpha_2 = m\alpha_2'', \ \ldots, \ \alpha_\lambda = m\alpha_\lambda'',$$

from which $$V_i = (a_1^{\ \alpha_1''} a_2^{\ \alpha_2''} \ldots a_\lambda^{\ \alpha_\lambda''})^m.$$

Let $V_i = V_h^{\ m}$. Then the equation of the curve becomes

$$(V_h y + V_{h+1})^m + k V_h^{\ m} \cdot V_h^{\frac{m}{h}} = 0.$$

Let $\dfrac{m}{h} = \dfrac{p}{q}$, where p and q are relatively prime. Then $V_h^{\frac{1}{q}}$ is rational since $V_h^{\frac{p}{q}}$ is rational. Hence, $p = 1$ since, otherwise, $f(x, y, t)$ is rationally factorable. Then $h = mq$ and $V_h^{\frac{1}{q}} = V_m$. The equation of the curve now takes the final form

$$(V_m^{\ q} y + V_{mq+1})^m + k V_m^{\ mq+1} = 0.$$

The locus of a point whose first polar has a line through $(0, 1, 0)$ as a component is the rational curve of order (in general) $mq + m - 1$ with an mq-fold point at $(0, 1, 0)$ defined parametrically by the equations

$$x' = V_{mt} V_m^{\ q}, \quad y' = V_{mx} V_{(mq+1)t} - V_{mt} V_{(mq+1)x}, \quad t' = -V_{mx} V_m^{\ q}.$$

ON THE ANALOGUE OF THE STUYVAERT GROUP OF CONGRUENCES OF CURVES AMONGST COMPLEXES OF CURVES IN SPACE OF FOUR DIMENSIONS.

By *C. G. F. James*, The University, Liverpool.

A CURVE in four dimensions may be given by a matrix of k rows and $k+2$ columns, whose elements are homogeneous polynomials in variables $x_1, \ldots, x_5$. If we allow these elements to be simultaneously functions of parameters $\alpha_1, \ldots, \alpha_4$, we obtain the representation of a complex or system ∞^3. A finite number of curves pass through an arbitrary point of the space, and this number is by definition the order of the system. It is, in our case, the number of solutions of the matrix for a fixed set of values of the (x), which are variable with the (x). It is therefore in general independent of the forms of the elements as functions of (x). The problem proposed is to find *the types for which the* (α) *enter at most linearly into the elements of the matrix, and which represent linear systems as defined above.* The analogous discussion for congruences of curves in ordinary space was given by Stuyvaert.*

We may regard the parameters as defining a space S_α of three dimensions, in which the matrix will represent a group of points, of which one only varies as (x) varies. It may also vanish for a curve or curves, but such curves must certainly be fixed as (x) varies. In our case the possibilities are:

I. *The separated determinants of the matrix represent* (*in* S_α) *planes*, which are not subject to any conditions. We denote the elements with such symbols as (i), (ijk), etc., the numbers in the brackets being the suffixes of the parameters the term contains. Elements from which the parameters are missing are denoted by $(-)$. The first case then gives rise to the following important cases:

$$\text{(A)} \quad \left\| \begin{matrix} (-) & (1\ 2\ 3\ 4) & (1\ 2\ 3\ 4) & (1\ 2\ 3\ 4) \\ (-) & (1\ 2\ 3\ 4) & (1\ 2\ 3\ 4) & (1\ 2\ 3\ 4) \end{matrix} \right\| = 0,$$

$$\text{(B)} \quad \left\| \begin{matrix} (1\ 2\ 3\ 4) & (1\ 2\ 3\ 4) & (1\ 2\ 3\ 4) & (1\ 2\ 3\ 4) \\ (-) & (-) & (-) & (-) \end{matrix} \right\| = 0.$$

* "Congruences de triangles, cubiques, etc.", *Crelle's Journal*, Bd. 132, p. 216.

In the remaining cases the separated determinants representing quadrics have a line in common whose points do not satisfy the matrix equation. In case II. the quadrics cut again in four points, of which three must be fixed. These are taken as the vertices $A_1A_2A_3$ of the tetrahedron of reference, and for each point the matrix must vanish identically, *i.e.* not in virtue of special values of the (x). These conditions are expressed exactly as in the theory for ordinary space.*

In case III. two of the three fixed points coincide (say) at the vertex A_4 with limiting direction A_4A_1, and the remaining point is taken as A_2. In case IV. the quadrics have at the vertex A_4 a common tangent plane (say) $A_4A_2A_3$. In each case the conditions may be expressed without difficulty. We do not tabulate the cases arising from II.–IV. separately, as there is a certain amount of overlapping in the results. Taking all possible types satisfying the various imposed conditions, and rejecting such as are mere special cases of others, we have finally as *the complete* solution of the problem proposed—

From case II.:

$$\text{(C)} \quad \left\|\begin{matrix} (1\ 2\ 4) & (1\ 2\ 4) & (1\ 2\ 4) & (1\ 2\ 4) \\ (3\ 4) & (3\ 4) & (3\ 4) & (3\ 4) \end{matrix}\right\| = 0,$$

$$\text{(D)} \quad \left\|\begin{matrix} (1\ 2\ 4) & (1\ 2\ 4) & (1\ 2\ 4) & (1\ 2\ 4) \\ (4) & (4) & (4) & (3\ 4) \end{matrix}\right\| = 0,$$

$$\text{(E)} \quad \left\|\begin{matrix} (2\ 3\ 4) & (2\ 3\ 4) & (2\ 3\ 4) & (2\ 3\ 4) \\ (1\ 3\ 4) & (1\ 3\ 4) & (1\ 3\ 4) & (1\ 3\ 4) \end{matrix}\right\| = 0,$$

where in (E) the coefficients of α_3 are the same for each pair of elements in the same column.

From case III.:

$$\text{(F)} \quad \left\|\begin{matrix} (1\ 2\ 3) & (1\ 2\ 3) & (1\ 2\ 3) & (1\ 2\ 3\ 4) \\ (1) & (1) & (1) & (1\ 2\ 3) \end{matrix}\right\| = 0.$$

From case IV.:

$$\text{(G)} \quad \left\|\begin{matrix} (3) & (1\ 3) & (1\ 3) & (1\ 3) \\ (1\ 2\ 3) & (1\ 2\ 3\ 4) & (1\ 2\ 3\ 4) & (1\ 2\ 3\ 4) \end{matrix}\right\| = 0,$$

$$\text{(H)} \quad \left\|\begin{matrix} (2\ 3) & (1\ 2\ 3) & (1\ 2\ 3) & (1\ 2\ 3) \\ (1\ 3) & (1\ 3\ 4) & (1\ 3\ 4) & (1\ 3\ 4) \end{matrix}\right\| = 0,$$

* Stuyvaert, *loc. cit.*

$$\text{(I)}\quad \left\|\begin{matrix} (3) & (1\ 3) & (1\ 3) & (1\ 2\ 3) \\ (1\ 3) & (1\ 3\ 4) & (1\ 3\ 4) & (1\ 2\ 3\ 4) \end{matrix}\right\| = 0,$$

$$\text{(J)}\quad \left\|\begin{matrix} (3) & (1\ 3) & (1\ 3) & (1\ 3\ 4) \\ (2\ 3) & (1\ 2\ 3) & (1\ 2\ 3) & (1\ 2\ 3\ 4) \end{matrix}\right\| = 0,$$

$$\text{(K)}\quad \left\|\begin{matrix} (1\ 3) & (1\ 3) & (1\ 3) & (1\ 3\ 4) \\ (2\ 3) & (1\ 2\ 3) & (1\ 2\ 3) & (1\ 2\ 3) \end{matrix}\right\| = 0,$$

$$\text{(L)}\quad \left\|\begin{matrix} (1\ 3) & (1\ 3) & (2\ 3) & (1\ 3\ 4) \\ (1\ 3) & (1\ 3) & (2\ 3) & (1\ 3\ 4) \end{matrix}\right\| = 0,$$

$$\text{(M)}\quad \left\|\begin{matrix} (1\ 3) & (1\ 3) & (1\ 3) & (2\ 3) \\ (1\ 3) & (1\ 3) & (1\ 3\ 4) & (1\ 2\ 3) \end{matrix}\right\| = 0.$$

There is one possibility with which we have not dealt. This is the case when the separated determinants represent quadrics, which, in addition to the line variable with the (x), share a fixed line and a fixed point, leaving one intersection which varies with the (x). Indeed, it is at once seen that the conditions imposed include those for case II., so that the resulting solutions will be included amongst those given.

END OF VOL. LIII.

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